ALGEBRAIC ANALYSIS OF SOME STRONGLY CLEAN RINGS AND THEIR GENERALIZATIONS

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Algebraic Analysis of Some Strongly Clean Rings and Their Generalizations

by

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Abstract

Let R be an associative ring with identity and $\mathrm{U}(R)$ denote the set of units of R. An element $a \in R$ is called clean if $a = e + u$ for some $e^2 = e$ and $u \in \mathrm{U}(R)$ and a is called strongly clean if, in addition, $eu = ue$. The ring R is called clean (resp., strongly clean) if every element of R is clean (resp., strongly clean). Let $\mathrm{Z}(R)$ be the center of R and $g(x)$ be a polynomial in the polynomial ring $\mathrm{Z}(R)[x]$. An element $a \in R$ is called $g(x)$-clean if $a = s + u$ where $g(s) = 0$ and $u \in \mathrm{U}(R)$ and a is called strongly $g(x)$-clean if, in addition, $su = us$. The ring R is called $g(x)$-clean (resp., strongly $g(x)$-clean) if every element of R is $g(x)$-clean (resp., strongly $g(x)$-clean). A ring R has stable range one if $Ra + Rb = R$ with $a, b \in R$ implies that $a + yb \in \mathrm{U}(R)$ for some $y \in R$.

In this thesis, we consider the following three questions:

- Does every strongly clean ring have stable range one?
- When is the matrix ring over a strongly clean ring strongly clean?
- What are the relations between clean (resp., strongly clean) rings and $g(x)$-clean (resp., strongly $g(x)$-clean) rings?

In the process of settling these questions, we actually get:

- The ring of continuous functions $\mathrm{C}(X)$ on a completely regular Hausdorff space X is strongly clean iff it has stable range one;
- A unital C^*-algebra with every unit element self-adjoint is clean iff it has stable range one;

- Necessary conditions for the matrix rings $\mathbb{M}_n(R)$ $(n \geq 2)$ over an arbitrary ring R to be strongly clean;
- Strongly clean property of $\mathbb{M}_2(RC_2)$ with certain local ring R and cyclic group $C_2 = \{1, g\}$;
- A sufficient but not necessary condition for the matrix ring over a commutative ring to be strongly clean;
- Strongly clean matrices over commutative projective-free rings or commutative rings having ULP;
- A sufficient condition for $\mathbb{M}_n(\mathrm{C}(X))$ $(\mathbb{M}_n(\mathrm{C}(X, \mathbb{C})))$ to be strongly clean;
- If R is a ring and $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ with $a, b \in \mathrm{Z}(R)$, then R is $(x-a)(x-b)$-clean iff R is clean and $b - a \in \mathrm{U}(R)$, and consequently, R is $g(x)$-clean when R is clean and $b - a \in \mathrm{U}(R)$;
- If R is a ring and $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ with $a, b \in \mathrm{Z}(R)$, then R is strongly $(x-a)(x-b)$-clean iff R is strongly clean and $b - a \in \mathrm{U}(R)$, and consequently, R is strongly $g(x)$-clean when R is strongly clean and $b - a \in \mathrm{U}(R)$.

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Contents

List of symbols

R	associative ring with identity
M_R	unitary right R-module
$_RM$	unitary left R-module
$\mathsf{mod}R$	category of right modules over a ring R
$\mathrm{End}(M_R)$	endomorphism ring of a right R-module M
$\mathrm{U}(R)$	group of multiplicative units in a ring R
$\mathrm{Z}(R)$	center of a ring R
$J(R)$	Jacobson radical of a ring R
$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\oplus_{i\in I}A_i$	direct sum of modules or other algebraic systems
$\Pi_{i\in I}A_i$	direct product of modules or other algebraic systems
$\mathbb{M}_n(R)$	$n \times n$ matrix ring over a ring R
$\mathbb{T}_n(R)$	$n \times n$ upper triangular matrix ring over a ring R
$GL_n(R)$	general linear group over R or multiplicative group of units in $\mathbb{M}_n(R)$

$R[x]$	polynomial ring over R in indeterminate x
$R[[x]]$	formal power series ring over R in indeterminate x
PID	principal ideal domain
UFD	unique factorization domain
$\deg(f(x))$	degree of the polynomial $f(x)$
$\gcd(f(x), g(x))$	greatest common divisor of $f(x)$ and $g(x)$ in the UFD $R[x]$ and it is monic if R is a field
$\chi_A(t)$	characteristic polynomial of a matrix A
$\det A$	determinant of a matrix A
$\|\cdot\|$	norm
$\mathrm{card}(\cdot)$	cardinal number
$\mathrm{char}(\cdot)$	character
$\mathfrak{m}$	maximal ideal of a ring
$\mathfrak{p}$	prime ideal of a ring
$\mathrm{Max}(R)$	maximal spectrum of maximal ideals in a commutative ring R
$\mathbb{Z}_{(p)}$	localization of $\mathbb{Z}$ at a prime ideal (p)
iff	if and only if

Introduction

A topological space X is said to be **completely regular** if whenever F is a closed set and x is a point in its complement, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(F) = \{0\}$. Let $\mathrm{C}(X)$ (resp., $\mathrm{C}(X, \mathbb{C})$) denote the ring of all real (resp., complex) valued continuous functions from a completely regular Hausdorff space X to the field of real numbers $\mathbb{R}$ (resp., field of complex numbers $\mathbb{C}$). For a function $f \in \mathrm{C}(X)$ (resp., $\mathrm{C}(X, \mathbb{C})$), the set $z(f)= \{x \in X : f(x) = 0\}$ is called the **zero-set** of f. An open set $U \subseteq X$ is called **functionally open** if the complement $X \backslash U$ is a zero-set. A topological space X is called **strongly zero-dimensional** if X is a completely regular T_1-space and every finite functionally open cover $\{U_i\}_{i=1}^{k}$ of the space X has a finite open refinement $\{V_i\}_{i=1}^{m}$ such that $V_i \cap V_j = \emptyset$ for any $i \neq j$ [26]. $\mathrm{C}(X)$ is a subject of general topology whose ring properties are intertwined intimately with the topological properties of X.

Let A be an algebra over $\mathbb{C}$. A self-map $*$ of A, $x \mapsto x^*$, is called an **involution** if it satisfies the following conditions: $x^{**} = x, (x + y)^* = x^* + y^*, (\alpha x)^* = \overline{\alpha} x^*$, and $(xy)^* = y^* x^*$. A Banach algebra A (see e.g. [75]) with the involution $*$ is said to be a **C*-algebra** whenever the norm $\|\cdot\|$ on A enjoys the equation $\|x^* x\| = \|x\|^2$ for all $x \in A$. When a C*-algebra A admits an identity, we call it a **unital C*-algebra**. An element $x \in A$ is **self-adjoint** provided $x = x^*$. The self-adjoint part of A is denoted by A_{sa}. A unital C*-algebra is said to **have real rank zero** if the set of invertible self-adjoint elements is dense in A_{sa} [9]. The C*-algebras with real rank zero form an important topic in non-commutative geometry.

Let R be an associative ring with identity and $\mathrm{U}(R)$ denote the set of units of R. Bass introduced the concept of stable range of a ring in [7]. For any $n \in \mathbb{N}, R^n$ denotes the R-module of all columns $b = (b_i)_{1 \le i \le n}$ over R and $U_{m_n}(R)$ is the subset of all unimodular columns in R^n, where a column $(b_i)_{1 \le i \le n}$ is **unimodular** if $Rb_1 + Rb_2 + \ldots + Rb_n = R$. R is said to satisfy the **n-stable range condition**, say $(1)_n$, if for every $(b_i)_{1 \le i \le n+1}$ in $U_{m_{n+1}}(R)$, there exists $(c_i)_{1 \le i \le n} \in R^n$ such that $(b_i + c_i b_{n+1})_{1 \le i \le n} \in U_{m_n}(R)$. The **stable range** (or **Bass' lowest stable range**) of R, $sr(R)$, is defined to be the smallest n such that $(1)_n$ holds. In particular, a ring R has **stable range one** if for $a, b \in R$ with $Ra + Rb = R$, there exists $y \in R$ such that $a + yb$ is left invertible. By [67, Theorem 2], this definition is left-right symmetric. In [68, Theorem 2.6], Vaserstein proved that, in a ring with stable range one, one-sided inverses are two-sided, so $a + yb \in \mathrm{U}(R)$ in definition. Equivalently, R has stable range one iff for any $a, x, b \in R$ satisfying $xa + b = 1$, there exists $y \in R$ such that $a + yb \in \mathrm{U}(R)$ [17, Proposition 1.1]. For exchange rings, R has stable range one iff for any $a, x, e^2 = e \in R$ satisfying $xa + e = 1$, there exists $y \in R$ such that $a + ye \in \mathrm{U}(R)$ [14, Lemma 2]. It is proved in [67, Theorem 3] that R has stable range one iff $\mathbb{M}_n(R)$ has stable range one for any $n \in \mathbb{N}$. Furthermore, if R has stable range one, then eRe has stable range one for any $e^2 = e \in R$ [68, Theorem 2.8]. It is also known that R has stable range one iff so does its opposite ring R_{op} iff so does $R/J(R)$ [67]. Bass showed that fields, division rings, semilocal rings, and artinian rings (in particular, finite-dimensional algebras) have stable range one [7]. For other examples, see [33, 68, 72]. It has been realized that the concept of stable range one is very useful in treating the stabilization problems in algebraic K-theory (cf. [50]).

Following von Neumann, an element a in a ring R is **regular** if $a = aba$ for some $b \in R$ and R is **regular** if every element is regular [54]. An element a in a ring R is **strongly regular** if $a = a^2x$ and $ax = xa$ for some $x \in R$ [4], or equivalently, $a = eu = ue$ for some $e^2 = e \in R$ and $u \in \mathrm{U}(R)$. R is **strongly regular** if every element is strongly regular. McCoy generalized regular rings to π-regular rings: An element $a \in R$ is **π-regular** if a^n is regular for some $n \in \mathbb{N}$ and R is **π-regular** if every element is π-regular [47].

Arens-Kaplansky [4] and Kaplansky [42] investigated π-regular rings and rings in which for every element a, the chain $aR \supseteq a^2R \supseteq \cdots$ terminates, that is, $a^n = a^{n+1}x$ for some $x \in R$ and some $n \in \mathbb{N}$. Azumaya [6] called such an element **right π-regular** and called a ring R **right π-regular** if every element is right π-regular. **Left π-regular elements** and **left π-regular rings** are defined analogously. An element is **strongly π-regular** if it is both left and right π-regular and R **strongly π-regular** if every element is strongly π-regular [6] (surprisingly, this is equivalent to saying that the chain $aR \supseteq a^2R \supseteq \cdots$ terminates for all $a \in R$ by Dischinger [23]). In [6], Azumaya proved that if $a \in R$ is strongly π-regular, then $a^n = a^{n+1}x$ and $ax = xa$ for some $x \in R$. So every strongly π-regular ring is π-regular. Strongly π-regular rings form an important class of rings which are generalizations of strongly regular rings, artinian rings, and perfect rings. Ara proved a nice result that every strongly π-regular ring has stable range one [2].

In 1964, Crawley and Jónsson introduced the well-known exchange property when they worked on direct sum refinements for algebraic systems: For a cardinality τ, a left R-module M is said to have the **τ-exchange property** if for any module X and decompositions $X = M' \oplus Y = \oplus_{i\in I}N_i$ with $M' \cong M$ and $\mathrm{card}(I) \leq \tau$, there exist submodules $N_i' \subseteq N_i$ for each i such that $X = M' \oplus (\oplus_{i\in I}N_i')$; If this condition holds for any set I, the module M is said to have the **(full) exchange property**; And if this condition holds for any finite set I, the module M is said to have the **finite exchange property** [22]. Warfield defined that a ring R is an **exchange ring** if ${}_RR$ has the finite exchange property [71]. An important result states that a unital C*-algebra A has real rank zero iff A is an exchange ring [3, Theorem 7.2]. Exchange theory has been one of the main research topics for ring theorists since it appeared in [71]. In 1977, Nicholson defined clean rings when he focused on exchange rings [57] (later, he defined strongly clean rings [58]). An element $a \in R$ is called **clean** if $a = e + u$ for some $e^2 = e$ and $u \in \mathrm{U}(R)$ and a is called **strongly clean** if, in addition, $eu = ue$. The ring R is called **clean** (resp., **strongly clean**) if every element of R is clean (resp., strongly clean). He proved that clean rings are exchange rings and that rings with central idempotents are clean iff they are exchange rings [57] (note that clean rings form a proper class of exchange rings by a counterexample of Bergman [37]). Clean and strongly clean rings have been extensively investigated, results and problems are surveyed in [61, 77]. A recent result about $C(X)$ is that X is strongly zero-dimensional iff $C(X)$ is clean iff it is strongly clean iff it is exchange [5, 49]. Strongly π-regular elements are strongly clean [58]. So strongly π-regular rings are strongly clean [10, 58]. In [58], Nicholson disclosed that strongly clean rings are a natural generalization of strongly π-regular rings by showing that $\alpha \in \mathrm{End}(M_R)$ is strongly π-regular iff $M_R = P_R \oplus Q_R$ such that $\alpha|_P : P \to P$ is an isomorphism and $\alpha|_Q : Q \to Q$ is nilpotent and that α is strongly clean iff $M_R = P_R \oplus Q_R$ such that $\alpha|_P : P \to P$ and $(1-\alpha)|_Q : Q \to Q$ are isomorphisms. Because Ara [2] proved a nice result that every strongly π-regular ring has stable range one, Nicholson asked:

Question 1: Does every strongly clean ring have stable range one?

In Chapter 1, motivated by the relation between clean rings and exchange rings and the relation between exchange rings and unital C*-algebras with real rank zero, we focus on $C(X)$ and some unital C*-algebras for cleanness and the property of stable range one, giving more affirmative examples to Question 1.

Let p be a property of modules that is preserved by isomorphism. Then p is called a **Morita invariant** if, for every additive equivalence $F : \mathsf{mod}R \to \mathsf{mod}S$, FX has p whenever X has p , this is equivalent to saying that whenever a ring R has p, whether or not the matrix ring $\mathbb{M}_n(R)$ has p as well as the corner ring eRe with $e^2 = e$ and $ReR = R$ [60, Theorem A.20]. For example, the properties of being stable range one, exchange, artinian, noetherian, semiperfect, and perfect are all Morita invariants. In his foundational paper [58], Nicholson asked whether or not strongly clean property is a Morita invariant. If R is strongly clean, then the corner ring eRe with $e^2 = e$ is strongly clean [66, 20, 18]. However, for a strongly clean ring R, the matrix ring $\mathbb{M}_n(R)$ need not be strongly clean [66, 69].

Question 2: When is the matrix ring over a strongly clean ring strongly clean?

Many authors dealt with this question [8, 19, 27, 45, 78, 79]. For local rings, Chen, Yang, and Zhou [19] characterized when the 2×2 matrix ring $\mathbb{M}_2(R)$ over a commutative local ring R is strongly clean; Li [45] characterized when a single 2×2 matrix over a commutative local ring is strongly clean; Fan and Yang [27] characterized when the matrix ring $\mathbb{M}_n(R)$ ($n = 3, 4$) over a commutative local ring R is strongly clean; Borooah, Diesl, and Dorsey [8] characterized when the matrix ring $\mathbb{M}_n(R)$ over a commutative local ring R is strongly clean; And recently, Yang and Zhou [79] characterized when the matrix ring $\mathbb{M}_2(R)$ over a local ring R is strongly clean. For strongly π-regular rings, Yang and Zhou [78] proved that the matrix rings over some strongly π-regular rings are strongly clean.

In Chapter 2, we get some partial answers to Question 2 when the underlying ring is a group ring, a commutative ring, a projective-free ring or a commutative ring having ULP (see Definition 2.4.5).

Let $\mathrm{Z}(R)$ be the center of R and $g(x)$ be a polynomial in the polynomial ring $\mathrm{Z}(R)[x]$. Following Camillo and Simón [15], an element $a \in R$ is called **$g(x)$-clean** if $a = s + u$ where $g(s) = 0$ and $u \in \mathrm{U}(R)$. R is called **$g(x)$-clean** if every element of R is $g(x)$-clean. They proved that if V is a countable-dimensional vector space over a division ring D and if $g(x) \in \mathrm{Z}(D)[x]$ has two distinct roots in $\mathrm{Z}(D)$, then $\mathrm{End}(V_D)$ is $g(x)$-clean [15]. Nicholson and Zhou generalized Camillo and Simón's result by proving that $\mathrm{End}({}_RM)$ is $g(x)$-clean where ${}_RM$ is a semisimple module over an arbitrary ring R and $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ with $a, b \in \mathrm{Z}(R)$ and b, $b - a \in \mathrm{U}(R)$ [62].

Question 3: What are the relations between clean rings and $g(x)$-clean rings?

In Chapter 3, we completely answer Question 3. In addition, we define strongly $g(x)$-clean rings (see Definition 3.2.1), and explore the relations between strongly $g(x)$-clean rings and strongly clean rings.

All given topological spaces are completely regular and Hausdorff unless specified.

Chapter 1

Strongly Clean Property vs Stable Range One

This chapter comprises two parts adapted from [29].

In Section 1.1, we prove that the ring of complex valued continuous functions $C(X, \mathbb{C})$ is strongly clean iff the ring of bounded complex valued continuous functions $C^*(X, \mathbb{C})$ is strongly clean iff X is strongly zero-dimensional. We also characterize when $C(X)$ is semilocal or local in this section.

In Section 1.2, focusing on the open problem asking whether or not strongly clean rings have stable range one [58], we prove that $C(X)$ is strongly clean iff it has stable range one, and a unital C^*-algebra in which every unit element is self-adjoint is clean iff it has stable range one. These results give more affirmative examples to the problem.

1.1 Strongly clean property of $C(X, \mathbb{C})$

We characterize when $C(X, \mathbb{C})$ is strongly clean using method of [5]. The identity of $C(X, \mathbb{C})$ is 1 with $1(x) = 1$ for all $x \in X$. Clearly, given an idempotent $e \in C(X, \mathbb{C})$, $z(e)$ is a clopen set, and given a clopen set $U \subseteq X$, we can get an idempotent e such that

$z(e) = U$.

Lemma 1.1.1 *$f \in C(X, \mathbb{C})$ is clean iff there exists a clopen set U in X such that $f^{-1}(\{1\}) \subseteq U \subseteq X \backslash z(f)$.*

Proof $\Rightarrow$ Suppose $f \in C(X, \mathbb{C})$ is clean, then $f = e + u$ with $e^2 = e \in C(X, \mathbb{C})$ and $u \in U(C(X, \mathbb{C}))$. Notice that $e^2 = e$ implies $e(x) = 0$ or $e(x) = 1$ for any $x \in X$. We can obtain that $f^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(f)$ with $z(e)$ a clopen set.

$\Leftarrow$ Suppose there is a clopen set U in X such that $f^{-1}(\{1\}) \subseteq U \subseteq X \backslash z(f)$.

Case 1. If f is invertible, then f is clean.

Case 2. If f is not invertible, then $z(f) \neq \emptyset$. Thus, U is a proper clopen set in X. Define $e : X \to \mathbb{C}$ by $e(x) = 0$ when $x \in U$ and $e(x) = 1$ when $x \in X \backslash U$. Then e is continuous and $e^2 = e$. Define $u : X \to \mathbb{C}$ by $u(x) = f(x)$ if $x \in U = z(e)$ and $u(x) = f(x) - 1$ if $x \in X \backslash U$. Then u is continuous. If $x \in U (\subseteq X \backslash z(f))$, then $f(x) \neq 0$, and hence, $u(x) \neq 0$. If $x \in X \backslash U$, then $f(x) \neq 1$, and hence, $u(x) = f(x) - 1 \neq 0$. Consequently, $u(x) \neq 0$ for any $x \in X$. It follows that u is invertible in $C(X, \mathbb{C})$. Clearly, $f = e + u$ is clean.

□

Lemma 1.1.2 *Let $f \in C^*(X, \mathbb{C})$. For each $\alpha \in \mathbb{R}$, set $A_\alpha = \{x \in X : |f(x)| \geq \alpha\}$. If there exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta < 1$ and a clopen set U in X such that $A_\beta \subseteq U \subseteq A_\alpha$, then f is clean in $C^*(X, \mathbb{C})$.*

Proof Let $U = z(e)$ for some $e^2 = e \in C(X, \mathbb{C})$. Define $u : X \to \mathbb{C}$ by $u(x) = f(x)$ for $x \in U$ and $u(x) = f(x) - 1$ for $x \in X \backslash U$. Then u is continuous. If $x \in U$, then $x \in A_\alpha$. So $|u(x)| = |f(x)| \geq \alpha$. If $x \in X \backslash U$, then $|u(x)| = |f(x) - 1| \geq 1 - |f(x)| \geq 1 - \beta$. So $u \in U(C^*(X, \mathbb{C}))$ and $f = e + u$ is clean in $C^*(X, \mathbb{C})$. □

Theorem 1.1.3 *The following are equivalent:*

(i) $\mathrm{C}(X, \mathbb{C})$ *is a (strongly clean, exchange) clean ring.*

(ii) $\mathrm{C}^*(X, \mathbb{C})$ *is a (strongly clean, exchange) clean ring.*

(iii) X *is strongly zero-dimensional.*

(iv) $\mathrm{C}(X)$ *is a (strongly clean, exchange) clean ring.*

(v) $\mathrm{C}^*(X)$ *is a (strongly clean, exchange) clean ring.*

Proof By [5, 49], it suffices to prove the equivalence of (i), (ii), and (iii).

(i) $\Rightarrow$ (ii) Given $f \in \mathrm{C}^*(X, \mathbb{C})$, let

$$A = \{x \in X : |f(x)| \geq \frac{2}{3}\} \text{ and } B = \{x \in X : |f(x)| \leq \frac{1}{3}\}.$$

Then A and B are two disjoint zero-sets and they are completely separated. Thus, there exists $g \in \mathrm{C}(X, \mathbb{C})$ such that $g(A) = \{1\}$ and $g(B) = \{0\}$. Since g is clean, there exists $e^2 = e \in \mathrm{C}(X, \mathbb{C})$ such that $g^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(g)$. By Lemma 1.1.2 and the inclusion chain $A = A_{\frac{2}{3}} \subseteq g^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(g) \subseteq X \backslash B \subseteq A_{\frac{1}{3}}$, we know that f is clean in $\mathrm{C}^*(X, \mathbb{C})$.

(ii) $\Rightarrow$ (iii) Let A and B be two completely separated subsets of X. Then there exists $f \in \mathrm{C}^*(X, \mathbb{C})$ such that $|f| \leq \frac{1}{2}$, $f(A) = \{0\}$, and $f(B) = \{\frac{1}{2}\}$. In this case, $f^{-1}(\{1\}) = \emptyset$ is a clopen set. So f is clean by Lemma 1.1.1. Again by Lemma 1.1.1, there exists $e^2 = e \in \mathrm{C}(X, \mathbb{C})$ such that $(2f)^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(2f)$. Hence, $B \subseteq (2f)^{-1}(\{1\}) \subseteq z(e) \subseteq X \backslash z(2f) \subseteq X \backslash A$. Since $z(e)$ is clopen, X is strongly zero-dimensional by [26, Theorem 6.2.4].

(iii) $\Rightarrow$ (i) Let $f \in \mathrm{C}(X, \mathbb{C})$ and $A = \{x \in X : f(x) = 1\}$. Clearly, A is a zero-set and $A \cap z(f) = \emptyset$. Hence, A and $z(f)$ are completely separated. Since X is strongly zero-dimensional, there exists a clopen set U such that $A \subseteq U \subseteq X \backslash z(f)$ [26, Lemma 6.2.2]. Thus, $f^{-1}(\{1\}) \subseteq A \subseteq U \subseteq X \backslash z(f)$. By Lemma 1.1.1, f is clean. $\square$

In the following, we will give some characterizations for $\mathrm{C}(X)$ to be semilocal or local. Recall that a ring R is called **semilocal** if $R/J(R)$ is semisimple. A commutative ring is semilocal iff it has finitely many maximal ideals.

If I is any ideal in $\mathrm{C}(X)$, then we define $Z[I] = \{z(f) : f \in I\}$. We call I a **fixed ideal** if $\cap Z[I] \neq \emptyset$, otherwise I is a **free ideal**. The symbol $I(r)$ denotes the residue class $r + I$.

Lemma 1.1.4 *[31, Theorem 4.6] The fixed maximal ideals in* $\mathrm{C}(X)$ *are precisely the sets* $M_p = \{f \in \mathrm{C}(X) : f(p) = 0\}$ $(p \in X)$*. The ideals* M_p *are distinct for distinct* p*. For each* p*,* $\mathrm{C}(X)/M_p$ *is isomorphic with* $\mathbb{R}$*. In fact, the mapping* $M_p(f) \mapsto f(p)$ *is the unique isomorphism from* $\mathrm{C}(X)/M_p$ *onto* $\mathbb{R}$*.*

Theorem 1.1.5 *Let* X *be a topological space. Then the following are equivalent:*

(i) $\mathrm{C}(X)$ *is a semisemple ring.*

(ii) $\mathrm{C}(X)$ *is a semilocal ring.*

(iii) $\mathrm{C}(X)$ *is clean and* $\mathrm{card}(X) < \infty$.

(iv) X *is strongly zero-dimensional and* $\mathrm{card}(X) < \infty$.

(v) X *is discrete with* $\mathrm{card}(X) < \infty$.

Proof (i) $\Leftrightarrow$ (ii) is obvious since $J(\mathrm{C}(X)) = 0$.

(ii) $\Rightarrow$ (iii) Suppose $M_1, M_2, \cdots, M_k$ are the maximal ideals of $\mathrm{C}(X)$. Then

$$\mathrm{C}(X) \cong \frac{\mathrm{C}(X)}{J(\mathrm{C}(X))} \cong \frac{\mathrm{C}(X)}{M_1 \cap M_2 \cap \cdots \cap M_k} \cong \frac{\mathrm{C}(X)}{M_1} \times \frac{\mathrm{C}(X)}{M_2} \times \cdots \times \frac{\mathrm{C}(X)}{M_k}$$

by the Chinese Remainder Theorem [1, p.103]. Because $\frac{\mathrm{C}(X)}{M_1}, \frac{\mathrm{C}(X)}{M_2}, \cdots, \frac{\mathrm{C}(X)}{M_k}$ are fields, $\frac{\mathrm{C}(X)}{M_1} \times \frac{\mathrm{C}(X)}{M_2} \times \cdots \times \frac{\mathrm{C}(X)}{M_k}$ is strongly clean. Thus $\mathrm{C}(X)$ is strongly clean. Since every point of X corresponds to a maximal ideal of $\mathrm{C}(X)$, we have $\mathrm{card}(X) < \infty$.

(iii) $\Rightarrow$ (iv) By [5, 49], $\mathrm{C}(X)$ is clean iff X is strongly zero-dimensional.

(iv) $\Rightarrow$ (v) This is because X is always Hausdorff.

(v) $\Rightarrow$ (ii) Since $\mathrm{card}(X) < \infty$, X is compact. So every ideal of $\mathrm{C}(X)$ is a fixed ideal [31, Theorem 4.8]. By Lemma 1.1.4, $\mathrm{C}(X)$ has finitely many maximal ideals. Thus $\mathrm{C}(X)$ is a semilocal ring. □

Corollary 1.1.6 *Let X be a topological space. Then the following are equivalent:*

(i) $\mathrm{C}(X)$ *is a local ring.*

(ii) $\mathrm{C}(X)$ *is a field isomorphic with* $\mathbb{R}$.

(iii) X *is singleton.*

Corollary 1.1.7 *Let X be a topological space with* $\mathrm{card}(X) < \infty$. *Then* $\mathbb{M}_n(\mathrm{C}(X))$ *is artinian.*

Proof By the proof of Theorem 1.1.5, $\mathrm{C}(X)$ is isomorphic with a finite direct product of fields. Hence $\mathrm{C}(X)$ is artinian. Thus $\mathbb{M}_n(\mathrm{C}(X))$ is artinian. □

1.2 Stable range one of $\mathrm{C}(X)$ and some unital C^*-algebras

Yu [81] proved that exchange rings with central idempotents have stable range one. So local rings and strongly regular rings have stable range one. Ara proved that strongly π-regular rings have stable range one [2]. In this section, we prove that $\mathrm{C}(X)$ and a unital C^*-algebra with every unit element self-adjoint are clean iff they have stable range one.

By the **order** of a family $\mathbb{A}$ of subsets of a set X, we mean the largest integer n such that the family $\mathbb{A}$ contains $n+1$ members with non-empty intersection, or the "infinite number" ∞ if no such integer exists. The order of a family $\mathbb{A}$ is denoted by ord$\mathbb{A}$. Let X be a space and let n denote an integer not less than -1, we say

(i) $dimX \leq n$ if every finite functionally open cover (consisting of non-empty functionally open sets) of X has a finite functionally open refinement of order $\leq n$.

(ii) $dimX = n$ if $dimX \leq n$ holds but $dimX \leq n-1$ does not hold.

(iii) $dimX = \infty$ if $dimX \leq n$ does not hold for any n.

The number $dimX$ is called the **covering dimension** of X [26, p.472]. By definition, we have $dimX = -1$ iff $X = \emptyset$, and $dimX = 0$ iff X is strongly zero-dimensional.

Theorem 1.2.1 *Let X be a topological space. Then* $\mathrm{C}(X)$ *is a clean ring iff* $\mathrm{C}(X)$ *has stable range one.*

Proof By [67, Theorem 5], $\mathrm{C}(X)$ has stable range one iff $dimX = 0$. By the definition of covering dimension, we know that $dimX = 0$ iff X is strongly zero-dimensional. So $\mathrm{C}(X)$ has stable range one iff $\mathrm{C}(X)$ is clean by [5, 49]. □

Corollary 1.2.2 *Let X be a strongly zero-dimensional space. Then* $\mathrm{C}(X)$ *has stable range one.*

Proof $\mathrm{C}(X)$ is strongly clean by [5, 49]. Thus $\mathrm{C}(X)$ has stable range one by Theorem 1.2.1. This corollary can be also obtained directly from [81]. □

Corollary 1.2.3 *If* $\mathrm{C}(X)$ *has stable range one, then* $\mathrm{C}(X, \mathbb{C})$ *has stable range one.*

Proof By Theorem 1.2.1, $\mathrm{C}(X)$ is a strongly clean ring. By [5, 49], X is strongly zero-dimensional, which is equivalent to saying that $dimX = 0$. So $\mathrm{C}(X, \mathbb{C})$ has stable range one by [67, Theorem 7]. □

In [3, Theorem 7.2], the authors proved that a unital C^*-algebra A has real rank zero iff A is an exchange ring. It is well known that all clean rings are exchange rings [57].

Thus, we want to know if a (strongly) clean unital C*-algebra has stable range one.

Theorem 1.2.4 *Let A be a unital* C*-*algebra in which every unit element is self-adjoint. Then A is clean iff A has stable range one.*

Proof $\Rightarrow$ Suppose A is clean. Let $r \in A$. By [13, Proposition 10], $r = u + v$ where $u \in \mathrm{U}(A)$ and $v^2 = 1$. Since $v = v^*$, we have $vv^* = v^*v = 1$, that is, r is a sum of a unitary and a unit. Thus, by [35, Theorem 4.1], A has stable range one.

$\Leftarrow$ Suppose A has stable range one. Let $r \in A$. By [35, Theorem 4.1], $r = u + v$ with $u \in \mathrm{U}(A)$ and $vv^* = v^*v = 1$. Because $v = v^*$, we have $v^2 = 1$. Therefore, A is clean by [13, Proposition 10]. □

It is an open problem whether or not the clean property of R implies that of the corner ring eRe [36]. However, for certain unital C*-algebras, we have the following positive answer.

Corollary 1.2.5 *Let A be a unital* C*-*algebra in which every element is self-adjoint. If A is clean, then eAe is clean for every $e^2 = e \in A$.*

Proof Let A be a unital C*-algebra in which every element is self-adjoint and A be clean. Then eAe is a unital C*-algebra in which every element is self-adjoint, and A has stable range one by Theorem 1.2.4. Since A has stable range one, it follows that eAe with $e^2 = e \in A$ has stable range one by [68, Theorem 2.8]. Again by Theorem 1.2.4, eAe is clean. □

We give an example (motivated by some examples in Zhu's textbook [82]) of a clean unital C*-algebra satisfying the condition of Theorem 1.2.4.

Example 1.2.6 *For a Hilbert space H, let $\mathrm{B}(H)$ denote the space of all bounded linear operators on H. If $T, L \in \mathrm{B}(H)$, define $TL = T \circ L$. With the usual adjoint operation as* *involution,* $\mathrm{B}(H)$ *becomes a unital* C^**-algebra. Let* $T \in \mathrm{B}(H)$ *be a self-adjoint operator and* A *be the* C^**-subalgebra generated by* T*. Then* A *is a unital* C^**-algebra in which every element is self-adjoint. It is clear that* A *is commutative. Moreover, the weak-operator closure of* A*, say* $\overline{A}$*, in* $\mathrm{B}(H)$ *is a commutative von Neumann algebra (a* C^**-subalgebra* $\mathbb{A}$ *of* $\mathrm{B}(H)$ *is called a* **von Neumann algebra** *if* $\mathbb{A}$ *is closed in the strong-operator topology) and* $\overline{A}$ *is a unital* C^**-algebra in which every element is self-adjoint. Every von Neumann algebra has real rank zero [46, Proposition 3.2.4]. Hence, it is an exchange ring. Nicholson proved that an exchange ring with central idempotents is clean [57]. Hence,* $\overline{A}$ *is (strongly) clean.*

Other examples confirm there exist unital C*-algebras with real rank zero which do not have stable range one and unital C*-algebras with stable range one which do not have real rank zero.

Example 1.2.7 (i) *Every purely infinite simple* C^**-algebra has real rank zero, but they do not have stable range one [9, Proposition 3.9]. Every von Neumann algebra has real rank zero [9, Proposition 1.3] [46, Proposition 3.2.4]. Pick a von Neumann algebra which is not finite, say* $\mathrm{B}(H)$ *with* H *an infinite-dimensional Hilbert space. Then* $\mathrm{B}(H)$ *has stable range infinity.*

(ii) *By [46, Proposition 3.1.3] [67, Theorem 7],* $\mathrm{C}([0,1],\mathbb{C})$ *has stable range one. However,* $\mathrm{C}([0,1],\mathbb{C})$ *does not have real rank zero by [9, Proposition 1.1].*

However, there are some relations between stable range one and real rank zero.

Example 1.2.8 (i) *Let* A *be a unital* C^**-algebra with real rank zero. Then* A *has stable range one iff* $eA \cong fA$ *with* $e^2 = e$ *and* $f^2 = f$ *implies* $e = ufu^{-1}$ *for some* $u \in \mathrm{U}(A)$*. In fact, if* A *is a unital* C^**-algebra with real rank zero, then* A *is an exchange ring by [3, Theorem 7.2]. This implication can be checked via [16, Lemma 1].*

(ii) *Let* A *be a unital* C^**-algebra in which every unit element is self-adjoint. If* A *has stable range one then* A *has real rank zero. Since if* A *has stable range one then*

Theorem 1.2.4 yields that A is clean, and consequently, A is an exchange ring. Thus A has real rank zero by [3, Theorem 7.2].

If X is a locally compact Hausdorff space, we say that a continuous function f from X to $\mathbb{C}$ **vanishes at infinity**, if for each positive number ϵ, the set $\{\omega \in X : |f(\omega)| \geq \epsilon\}$ is compact. We denote the set of such functions by $\mathrm{C}_0(X, \mathbb{C})$, which is a C*-algebra with involution $f \mapsto \overline{f}$ where $\overline{f}$ is the conjugate of f. It is unital iff X is compact, and in this case $\mathrm{C}_0(X, \mathbb{C}) = \mathrm{C}(X, \mathbb{C})$ [52]. It turns out for any space X, $\mathrm{C}(X, \mathbb{C})$ need not be a unital C*-algebra. However, we have the following result.

Corollary 1.2.9 *Let X be a compact topological space. If $\mathrm{C}(X, \mathbb{C})$ has real rank zero, then $\mathrm{C}(X, \mathbb{C})$ has stable range one.*

Proof By [3, Theorem7.2], $\mathrm{C}(X, \mathbb{C})$ has real rank zero iff $\mathrm{C}(X, \mathbb{C})$ is an exchange ring. Exchange rings with central idempotents have stable range one [81]. So $\mathrm{C}(X, \mathbb{C})$ has real rank zero implies that $\mathrm{C}(X, \mathbb{C})$ has stable range one. $\square$

Chapter 2

Strongly Clean Matrix Rings

This chapter consists of five sections in which the first two come from [30].

In Section 2.1, necessary conditions for $\mathbb{M}_n(R)$ $(n \geq 2)$ over an arbitrary ring R to be strongly clean are given.

In Section 2.2, the strongly clean property of $\mathbb{M}_2(RC_2)$ over the group ring RC_2 with certain local ring R and $C_2 = \{1, g\}$ is obtained.

In Section 2.3, we generalize the concept of SRC factorization [8] from commutative local rings to commutative rings, get a sufficient but not necessary condition for a matrix ring over a commutative ring to be strongly clean, and characterize the n-SRC rings.

In Section 2.4, we study the strong cleanness of matrices over a commutative projective-free ring or a commutative ring having ULP.

In Section 2.5, we prove that the matrix ring over either $C(X)$ or $C(X, \mathbb{C})$ is strongly π-regular when X is a P-space or P-space relative to $\mathbb{C}$.

2.1 Necessary conditions on strongly clean matrix rings

Chen, Yang, and Zhou [19, Theorem 5] gave a necessary condition for the matrix ring $\mathbb{M}_2(R)$ over a commutative ring R to be strongly clean. Dorsey [24, Theorem 3.7.2] gave a necessary condition for $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean. Here, we give necessary conditions for $\mathbb{M}_n(R)$ $(n \geq 2)$ over an arbitrary ring R to be strongly clean.

For a ring R and a polynomial $f(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n \in R[t]$, an element $r \in R$ is called a **left** (resp., **right**) **root** of $f(t)$ if $a_0 + ra_1 + r^2a_2 + \cdots + r^na_n = 0$ (resp., $a_0 + a_1r + a_2r^2 + \cdots + a_nr^n = 0$).

Lemma 2.1.1 *[76] Let R be a ring with $w_0, w_1 \in R$. Consider two polynomials $f(t) = t^2 - (1+w_0)t - w_1$ and $g(t) = t^2 - (1-w_0)t - (w_0 + w_1)$ over R. Then the following are equivalent for $t_0 \in R$:*

(i) *t_0 is a left root of $f(t)$.*

(ii) *$1 + w_0 - t_0$ is a right root of $f(t)$.*

(iii) *$1 - t_0$ is a left root of $g(t)$.*

Theorem 2.1.2 *Let R be a ring for which $\mathbb{M}_2(R)$ is strongly clean and let $f(t) = t^2 - (1+w_0)t - w_1$ be a polynomial with $w_0, w_1 \in J(R)$. Then the following hold:*

(i) *$f(t)$ has a right root in $J(R)$ and a right root in $1 + J(R)$.*

(ii) *$f(t)$ has a left root in $J(R)$ and a left root in $1 + J(R)$.*

Proof (i) Let $A = \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ and $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for R^2. Under this basis, A corresponds to $\varphi_A \in \text{End}((R^2)_R)$. For computation simplicity, we identify the matrix A with the corresponding endomorphism φ_A. It is clear that A

and $I-A$ are non-invertible. By [58], $(R^2)_R$ has a nontrivial $R_1R_2R_1R_2$-decomposition

$$\begin{array}{ccccc} (R^2)_R & = & R_1 & \oplus & R_2 \\ & & {\scriptstyle A}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle I-A}\big\downarrow{\scriptstyle\cong} \\ (R^2)_R & = & R_1 & \oplus & R_2 \end{array}$$

with $0 \neq R_1 < (R^2)_R$ and $0 \neq R_2 < (R^2)_R$. For notation convenience, in what follows let bar denote the natural epimorphisms. For example, the natural homomorphism $R \to \overline{R} = R/J(R)$ stands for $r \mapsto \overline{r} = r + J(R)$. Let $rad(R^2)$ be the Jacobson radical of the module $(R^2)_R$. Since $A : R_1 \to R_1$ is an isomorphism, we get an isomorphism $\overline{A} : (R_1 + rad(R^2))/rad(R^2) \to (R_1 + rad(R^2))/rad(R^2)$ with $\overline{A}(\overline{r}) = \overline{A(r)}$. Similarly, $\overline{I} - \overline{A} : (R_2 + rad(R^2))/rad(R^2) \to (R_2 + rad(R^2))/rad(R^2)$ is also an isomorphism. For $\overline{A}$ and $\overline{I} - \overline{A}$, we have

$$\overline{R_1} = \overline{A}(\overline{R_1}) \subseteq \overline{A}(\overline{R}^2) = \operatorname{Im}\begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix} = \overline{e}_1\overline{R} \tag{2.1.1}$$

and

$$\overline{R_2} = (\overline{I} - \overline{A})(\overline{R_2}) \subseteq (\overline{I} - \overline{A})(\overline{R}^2) = \operatorname{Im}\begin{pmatrix} \overline{0} & -\overline{1} \\ \overline{0} & \overline{1} \end{pmatrix} = (\overline{e}_2 - \overline{e}_1)\overline{R}. \tag{2.1.2}$$

Since $R_1 \oplus R_2 = R^2$, we have $\overline{R_1} \oplus \overline{R_2} = \overline{R}^2$. By $\overline{e}_1\overline{R} \oplus (\overline{e}_2 - \overline{e}_1)\overline{R} = \overline{R}^2$, (2.1.1), and (2.1.2), we get $\overline{R_1} = \overline{e}_1\overline{R}$ and $\overline{R_2} = (\overline{e}_2 - \overline{e}_1)\overline{R}$. Let $E : R^2 = R_1 \oplus R_2 \to R_1$ be the projection onto R_1 with kernel R_2. Then $I - E : R^2 = R_1 \oplus R_2 \to R_2$ is the projection onto R_2 with kernel R_1. Let $\eta_1 = Ee_2, \eta_2 = (I-E)e_2$. Then $\eta_1 \in R_1$ and $\eta_2 \in R_2$. So $Ee_2 + (I-E)e_2 = e_2 = e_1 + (e_2 - e_1)$. Hence, $\overline{E}\overline{e}_2 + (\overline{I} - \overline{E})\overline{e}_2 = \overline{e}_2 = \overline{e}_1 + (\overline{e}_2 - \overline{e}_1)$. Since $\overline{E}\overline{e}_2$ and $\overline{e}_1$ are in $\overline{R_1}$, $(\overline{I} - \overline{E})\overline{e}_2$ and $\overline{e}_2 - \overline{e}_1$ are in $\overline{R_2}$, and $\overline{R_1} \oplus \overline{R_2} = \overline{R}^2$, we get $\overline{\eta}_1 = \overline{E}\overline{e}_2 = \overline{e}_1$ and $\overline{\eta}_2 = (\overline{I} - \overline{E})\overline{e}_2 = \overline{e}_2 - \overline{e}_1$. So $\{\overline{\eta}_1, \overline{\eta}_2\} = \{\overline{E}\overline{e}_2, (\overline{I} - \overline{E})\overline{e}_2\} = \{\overline{e}_1, \overline{e}_2 - \overline{e}_1\}$ is a basis for $\overline{R}^2$. Then $\eta_1 R + \eta_2 R + rad(R^2) = R^2$. However, $rad(R^2)$ is superfluous in R^2, so we get $\eta_1 R + \eta_2 R = R^2$. That is, $\{\eta_1, \eta_2\}$ generates R^2 as a right R-module. Let $\eta_1 r_1 + \eta_2 r_2 = 0$. Then $\eta_1 r_1 = 0$ and $\eta_2 r_2 = 0$ because $\eta_1 \in R_1$ and $\eta_2 \in R_2$. Let $\eta_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\eta_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then, by $\overline{\eta}_1 = \overline{E}\overline{e}_2 = \overline{e}_1 = \begin{pmatrix} \overline{1} \\ \overline{0} \end{pmatrix}$ and $\overline{\eta}_2 = (\overline{I} - \overline{E})\overline{e}_2 = \overline{e}_2 - \overline{e}_1 = \begin{pmatrix} -\overline{1} \\ \overline{1} \end{pmatrix}$, we get $x_1, y_2 \in 1 + J(R)$, $y_1 \in J(R)$, and $x_2 \in -1 + J(R)$. So $r_1 = 0$ by $\eta_1 r_1 = \begin{pmatrix} x_1 r_1 \\ y_1 r_1 \end{pmatrix} =$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r_2 = 0$ by $\eta_2 r_2 = \begin{pmatrix} x_2 r_2 \\ y_2 r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So η_1 and η_2 are R-linearly independent. Hence, $\{\eta_1, \eta_2\}$ is a basis for R^2. If $r_1 \in R_1$ such that $r_1 = \eta_1 l_1 + \eta_2 l_2$ with $l_1, l_2 \in R$, then $(r_1 - \eta_1 l_1) - \eta_2 l_2 = 0$. Hence, $r_1 - \eta_1 l_1 = 0$ and $\eta_2 l_2 = 0$. So $r_1 = \eta_1 l_1$ and l_1 is uniquely determined because $r_1 = \begin{pmatrix} x_1 l_1 \\ y_1 l_1 \end{pmatrix}$ with $x_1 \in \mathrm{U}(R)$. So η_1 is a basis for R_1. Similarly, $R_2 = \eta_2 R$ is free with basis η_2. Let $\eta_1' = \eta_1 x_1^{-1} = \begin{pmatrix} 1 \\ x \end{pmatrix}$ with $x = y_1 x_1^{-1} \in J(R)$. Then η_1' is also a basis for R_1. Now $A : R_1 \to R_1$ is an isomorphism. Let $A\eta_1' = \eta_1' r$ with $r \in R$. That is, $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} r$. So $\begin{pmatrix} 1+w_0+x \\ w_1 \end{pmatrix} = \begin{pmatrix} r \\ xr \end{pmatrix}$. Hence, $r = 1 + w_0 + x$. Notice that

$$\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}(1+w_0) - Iw_1 = 0.$$

So

$$\begin{aligned} &\left(\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}(1+w_0) - Iw_1 \right)(\eta_1') \\ &= A^2\eta_1' - A(1+w_0)\eta_1' - w_1\eta_1' = 0. \end{aligned} \tag{2.1.3}$$

By a direct computation, we get

$$A^2\eta_1' = \eta_1'(1+w_0+x)^2 = \begin{pmatrix} 1 \\ x \end{pmatrix}(1+w_0+x)^2 = \begin{pmatrix} (1+w_0+x)^2 \\ x(1+w_0+x)^2 \end{pmatrix}, \tag{2.1.4}$$

$$A(1+w_0)\eta_1' = \begin{pmatrix} (1+w_0)^2 & 1+w_0 \\ w_1(1+w_0) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} (1+w_0)^2 + (1+w_0)x \\ w_1(1+w_0) \end{pmatrix}, \tag{2.1.5}$$

and

$$w_1\eta_1' = \begin{pmatrix} w_1 \\ w_1 x \end{pmatrix}. \tag{2.1.6}$$

Combining (2.1.3), (2.1.4), (2.1.5), and (2.1.6), we get

$$(1+w_0+x)^2 - (1+w_0)(1+w_0+x) - w_1 = 0.$$

Namely, $1 + w_0 + x \in 1 + J(R)$ is a right root of $t^2 - (1+w_0)t - w_1 = 0$. By Lemma 2.1.1, we know that $t^2 - (1+w_0)t - w_1 = 0$ also has a right root in $J(R)$.

(ii) The proof is similar to the above. $\square$

Corollary 2.1.3 *[24, Theorem 3.7.2] Let R be a ring for which $\mathbb{M}_2(R)$ is strongly clean. Then the polynomial t^2-t-w has a root in $J(R)$ and a root in $1+J(R)$ for any $w \in J(R)$.*

Corollary 2.1.4 *[19, Theorem 5] Let R be a commutative ring. If $\mathbb{M}_2(R)$ is strongly clean, then $t^2 - t = w$ is solvable in R for any $w \in J(R)$.*

When R is a local ring, many equivalent conditions of the strongly clean property of $\mathbb{M}_2(R)$ are given in [79] and one of them is that $\mathbb{M}_2(R)$ is strongly clean iff for any $w_0, w_1 \in J(R)$, the polynomial $t^2 - (1 + w_0)t - w_1$ has a right (or left) root in $J(R)$ and a right (or left) root in $1 + J(R)$. In other words, the conclusions of Theorem 2.1.2 are equivalent to the strongly clean property of $\mathbb{M}_2(R)$ when R is local. Moreover, when R is commutative local, the strongly clean property of $\mathbb{M}_2(R)$ is equivalent to the solvability of the equation $t^2 - t = w$ for any $w \in J(R)$ [8, 19, 45].

Theorem 2.1.5 *([66, 20, 18]) Let R be a strongly clean ring. Then, for any $e^2 = e \in R$, eRe is strongly clean.*

Corollary 2.1.6 *Let m, n be positive integers such that $m < n$. If the matrix ring $\mathbb{M}_n(R)$ is strongly clean, then so is $\mathbb{M}_m(R)$.*

Proof Let $e = \mathrm{diag}(1, \cdots, 1, 0, \cdots 0)$ be the diagonal matrix with (i,i)-th entry being 1 $(i = 1, \cdots, m)$ and other entries 0. Then $\mathbb{M}_m(R) \cong e\mathbb{M}_n(R)e$. □

By Corollary 2.1.6, we can see that if $\mathbb{M}_2(R)$ is not strongly clean then $\mathbb{M}_n(R)$ is not strongly clean for any $n \geq 2$. Thus when we want to determine the strongly clean property of a matrix ring, we should consider the strongly clean property of $\mathbb{M}_2(R)$ firstly.

Corollary 2.1.7 *For $1 < n \in \mathbb{N}$, let R be a ring such that $\mathbb{M}_n(R)$ is strongly clean and $f(t) = t^2 - (1 + w_0)t - w_1$ be a polynomial with $w_0, w_1 \in J(R)$. Then the following hold:*

(i) *$f(t)$ has a right root in $J(R)$ and a right root in $1 + J(R)$.*

(ii) *$f(t)$ has a left root in $J(R)$ and a left root in $1 + J(R)$.*

Proof This follows from Theorem 2.1.2 and Corollary 2.1.6. □

2.2 Strongly clean matrix ring $\mathbb{M}_2(RC_2)$

Motivated by [8, 19, 78] discussing the strongly clean property of group rings RG and matrix rings over RG with R commutative local or strongly π-regular, we deal with the strongly clean property of $\mathbb{M}_2(RC_2)$ where R is certain local ring and $C_2 = \{1, g\}$ is the cyclic group of order 2.

Definition 2.2.1 *[79] A local ring R is called* ***weakly bleached*** *if, for all $j_1, j_2 \in J(R)$, the additive abelian group endomorphisms of R, $l_{1+j_1} - r_{j_2} : x \mapsto (1 + j_1)x - xj_2$ and $l_{j_2} - r_{1+j_1} : x \mapsto j_2x - x(1 + j_1)$ are surjective.*

As shown in [79], there are many weakly bleached rings.

Lemma 2.2.2 *Let R be a local ring with $2 \in J(R)$. Then RC_2 is a local ring with $J(RC_2) = \{a + bg | a, b \in R, a + b \in J(R)\}$.*

Proof By Nicholson [55], RC_2 is local since R is local, C_2 is a abelian 2-group, and $2 \in J(R)$. Let $\nabla(RC_2) = \{a + bg | a, b \in R, a + b \in J(R)\}$. Then $\nabla(RC_2)$ is an ideal of RC_2. Thus to prove $J(RC_2) = \nabla(RC_2)$, it suffices to prove $J(RC_2) \subseteq \nabla(RC_2)$ since $J(RC_2)$ is the maximal ideal of RC_2. Note that local rings are directly finite.

Suppose $a+bg \in J(RC_2)$, then, for any $m+ng \in RC_2$, $1-(m+ng)(a+bg) \in \mathrm{U}(RC_2)$, that is, $(1 - ma - nb) - (mb + na)g \in \mathrm{U}(RC_2)$. So there exists $x + yg \in RC_2$ such that $((1 - ma - nb) - (mb + na)g)(x + yg) = 1$. Hence,

$$\begin{cases} (1 - ma - nb)x - (mb + na)y = 1 \\ (1 - ma - nb)y - (mb + na)x = 0. \end{cases}$$

So $(1 - ma - nb - (mb + na))\, x + (1 - ma - nb - (mb + na))\, y = 1$. That is,

$$(1 - ma - nb - (mb + na))\, (x + y) = 1.$$

Let $m+n=t$. Then $1-t(a+b)\in \mathrm{U}(R)$ for any $t\in R$. So $a+b\in J(R)$. Consequently, $J(RC_2)\subseteq \nabla(RC_2)$. The proof is complete. $\square$

Theorem 2.2.3 *Let R be a weakly bleached local ring with $2\in \mathrm{U}(R)$ or* $\mathrm{char}R=2$. *Then the following are equivalent:*

(i) $\mathbb{M}_2(R)$ *is strongly clean.*

(ii) $\mathbb{M}_2(RC_2)$ *is strongly clean.*

Proof (ii) $\Rightarrow$ (i) This is because $\mathbb{M}_2(R)$ is the homomorphic image of $\mathbb{M}_2(RC_2)$ and the homomorphic image of a strongly clean ring is strongly clean.

(i) $\Rightarrow$ (ii) **Case 1**. $2\in \mathrm{U}(R)$. Then $\alpha: RC_2\to R^2$ with $\alpha(a+bg)=(a+b,a-b)$ is a ring isomorphism. So $\mathbb{M}_2(RC_2)(\cong \mathbb{M}_2(R)\oplus\mathbb{M}_2(R))$ is strongly clean (in this case, (i)$\Rightarrow$(ii) holds without the condition of R being local or weakly bleached).

Case 2. $\mathrm{char}R=2$. We prove that $t^2-t(1+w_0)=w_1$ has a left root in $J(RC_2)$ where $w_0,w_1\in J(RC_2)$.

By Lemma 2.2.2, we suppose $t=x+(j_0-x)g$, $w_0=a+(j_1-a)g$, and $w_1=b+(j_2-b)g$ where x,j_0 are variables and $j_0,j_1,j_2\in J(R)$.

$$
\begin{aligned}
&t^2-t(1+w_0)=w_1\\
\Leftrightarrow\ &(x+(j_0-x)g)^2-(x+(j_0-x)g)(1+a+(j_1-a)g)=b+(j_2-b)g\\
\Leftrightarrow\ &\begin{cases} x^2+(j_0-x)^2-x(1+a)-(j_0-x)(j_1-a)=b\\ x(j_0-x)+(j_0-x)x-x(j_1-a)-(j_0-x)(1+a)=j_2-b\end{cases}\\
\Leftrightarrow\ &\begin{cases} j_0^2+j_0x+xj_0+x+j_0j_1+j_0a+xj_1=b\\ j_0x+xj_0+x+j_0+j_0a+xj_1=j_2-b\end{cases}\\
\Leftrightarrow\ &\begin{cases} j_0^2+j_0(1+j_1)=j_2\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)\\ x(1+j_0+j_1)+j_0x=b+j_0+j_0a+j_2\cdots\cdots\cdots(2).\end{cases}
\end{aligned}
\tag{2.2.1}
$$

By [79, Theorem 7], (2.2.1)(1) has a left root $j_0\in J(R)$. Then by (2.2.1)(2), we can find a right root $x\in R$ because R is weakly bleached. Again by [79, Theorem 7], $\mathbb{M}_2(RC_2)$ is strongly clean. $\square$

Theorem 2.2.4 *Let R be a local ring with $2 \in J(R)$ and* $\mathrm{char}R \neq 2$. *If the equation* $(*) : 2t^2 + a_0t + ta_1 + a_2 = 0$ *is solvable in R where $a_2 \in R$. $a_0 \in J(R)$, and $a_1 \in 1 + J(R)$, then the following are equivalent:*

(i) $\mathbb{M}_2(R)$ *is strongly clean.*

(ii) $\mathbb{M}_2(RC_2)$ *is strongly clean.*

Proof (ii) $\Rightarrow$ (i) Note that $\mathbb{M}_2(R)$ is the homomorphic image of $\mathbb{M}_2(RC_2)$.

(i) $\Rightarrow$ (ii) We prove that $t^2 - t(1 + w_0) = w_1$ has a left root in $J(RC_2)$ where $w_0, w_1 \in J(RC_2)$.

Suppose $t = x + (j_0 - x)g$, $w_0 = a + (j_1 - a)g$, and $w_1 = b + (j_2 - b)g$ where x, j_0 are variables and $j_0, j_1, j_2 \in J(R)$.

$$\begin{aligned}
& t^2 - t(1 + w_0) = w_1 \\
\Leftrightarrow\ & (x + (j_0 - x)g)^2 - (x + (j_0 - x)g)(1 + a + (j_1 - a)g) = b + (j_2 - b)g \\
\Leftrightarrow\ & \begin{cases} x^2 + (j_0 - x)^2 - x(1 + a) - (j_0 - x)(j_1 - a) = b \\ x(j_0 - x) + (j_0 - x)x - x(j_1 - a) - (j_0 - x)(1 + a) = j_2 - b \end{cases} \\
\Leftrightarrow\ & \begin{cases} j_0^2 + 2x^2 - j_0x - xj_0 - 2xa - x + xj_1 - j_0j_1 + j_0a = b \\ -2x^2 + j_0x + xj_0 + 2xa + x - xj_1 - j_0 - j_0a = j_2 - b \end{cases} \\
\Leftrightarrow\ & \begin{cases} j_0^2 - j_0(1 + j_1) = j_2 \cdots\cdots\cdots\cdots (1) \\ 2x^2 - j_0x - x(1 + j_0 + 2a - j_1) = b + j_0j_1 - j_0a - j_0^2 \cdots\cdots (2). \end{cases}
\end{aligned} \tag{2.2.2}$$

By [79, Theorem 7], (2.2.2)(1) has a left root $j_0 \in J(R)$ and by $(*)$ we can find a solution $x \in R$ to (2.2.2)(2). Then by [79, Theorem 7], $\mathbb{M}_2(RC_2)$ is strongly clean. $\square$

Corollary 2.2.5 *[19, Theorem 12] Let R be a commutative local ring. Then the following are equivalent:*

(i) $\mathbb{M}_2(R)$ *is strongly clean.*

(ii) $\mathbb{M}_2(RC_2)$ *is strongly clean.*

Proof Every commutative local ring is automatically weakly bleached. By Theorem 2.2.3 and Theorem 2.2.4, we need only to prove the solvability of $(*)$ in Theorem 2.2.4 when $\mathbb{M}_2(R)$ is strongly clean and $2 \in J(R)$.

Claim. The solvability of $(*)$ is equivalent to the solvability of $2t^2 - t = b$ with $b \in R$.

If $(*)$ is solvable, then $2t^2 + (1 + j_0 + j_1)t + a = 0$ is solvable for any $j_0, j_1 \in J(R)$ and $a \in R$. Take $j_1 = -2 - j_0$ and $a = -b$. Then $j_1 \in J(R)$ since $2 \in J(R)$ and $1 + j_0 + j_1 = -1$. Consequently, $2t^2 - t = b$ is solvable for any $b \in R$.

If $2t^2 - t = b$ is solvable for any $b \in R$, we want to show $(*)$ is solvable, that is, $2t^2 + (1 + j_0 + j_1)t + a = 0$ is solvable for any $j_0, j_1 \in J(R)$ and $a \in R$. Thus it suffices to show the solvability of $2t^2 + ut + a = 0$ for any $u \in \mathrm{U}(R)$ and $a \in R$. Set $t = -ux$. Then $2t^2 + ut + a = 2u^2x^2 - u^2x + a$. So the solvability of $2t^2 + ut + a = 0$ is equivalent to the solvability of $2x^2 - x + (u^2)^{-1}a = 0$ which is solvable by the solvability of $2t^2 - t = b$ for any $b \in R$.

When $\mathbb{M}_2(R)$ is strongly clean, $t^2 - t = 2b$ is solvable with a root $t_0 \in J(R)$ by [24, Theorem 3.7.2]. Hence, $b(t_0 - 1)^{-1}$ is a root of $2t^2 - t = b$. That is, when $\mathbb{M}_2(R)$ is strongly clean, $2t^2 - t = b$ with $b \in R$ is always solvable. So (i) is equivalent to (ii). $\square$

2.3 Strong cleanness of $\mathbb{M}_n(R)$ over a commutative ring R

The authors of [8] defined the so-called SR factorization and SRC factorization. To be more precise, let R be a commutative local ring. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an **SR factorization** if $h_0(t)$ and $h_1(t)$ are monic and $h_0(0)$ and $h_1(1) \in \mathrm{U}(R)$. The ring R is an n-**SR ring** if every monic polynomial of degree n in $R[t]$ has an SR factorization. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an **SRC factorization** if it is an SR factorization and gcd $(\overline{h}_0(t), \overline{h}_1(t)) = 1$ in the PID $\bar{R}[t]$ $(= \frac{R}{J(R)}[t])$. The ring R is an n-**SRC ring** if every monic polynomial of degree n in $R[t]$ has an SRC factorization. R is an **SRC ring** if it is an n-SRC ring for every $n \in \mathbb{N}$. They proved that for a commutative local ring R, $\mathbb{M}_n(R)$ is strongly clean iff R is an n-SRC ring and that a matrix ring over a Henselian ring [53] is strongly clean. The theory of SRC factorization is a useful tool for judging strong cleanness of matrix rings over commutative local rings. However, the theory is constraint to commutative local rings.

Because 0 and 1 are the only idempotents of local rings, we may generalize the above definition to commutative rings. Before making such a generalization, we recall that, for a commutative ring R, a pair of polynomials $(f_0(t), f_1(t))$ in $R[t]$ is **unimodular** provided $f_0(t)R[t] + f_1(t)R[t] = R[t]$, or equivalently, $f_0(t)h_0(t) + f_1(t)h_1(t) = 1$ with some $h_0(t)$ and $h_1(t)$ in $R[t]$, and for a commutative local ring R and monic polynomials $f_0(t)$ and $f_1(t)$ in $R[t]$, gcd $(\overline{f}_0(t), \overline{f}_1(t)) = 1$ iff $(\overline{f}_0(t), \overline{f}_1(t))$ is unimodular in $\bar{R}[t]$ iff $(f_0(t), f_1(t))$ is unimodular in $R[t]$.

Definition 2.3.1 *Let R be a commutative ring and $f(t) \in R[t]$ be a monic polynomial. A factorization $f(t) = f_0(t)f_1(t)$ in $R[t]$ is called an* **SR factorization** *if $f_i(t)$ is monic in $R[t]$ and $f_i(e_i) \in \mathrm{U}(R)$ with idempotents $e_0 \neq e_1 \in R$ $(i = 0, 1)$. The factorization $f(t) = f_0(t)f_1(t)$ is called an* **SRC factorization** *if, in addition, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. The ring R is called an n-**SR** (resp., n-**SRC**) **ring** if every monic polynomial of degree n has an* SR *(resp.,* SRC*) factorization.*

Theorem 2.3.2 *Let R be a commutative ring. Then R is strongly clean iff R is a* 1-SR *ring iff R is a* 1-SRC *ring.*

Proof Suppose R is strongly clean. Let $f(t) = t + a \in R[t]$. Write $-a = e + u$ where $e^2 = e \in R$, $u \in \mathrm{U}(R)$, and $eu = ue$. So $f(e) = -u \in \mathrm{U}(R)$. Hence, $f(t) = f_0(t)f_1(t)$ with $f_0(t) = t + a$ and $f_1(t) = 1$ is an SR factorization. Obviously, this is also an SRC factorization.

Suppose R is a 1-SR ring. Let $a \in R$. Then $f(t) = t - a$ has an SR factorization in

$R[t]$. It must be that $f(t) = f_0(t)$ or $f(t) = f_1(t)$. So there exists $e^2 = e \in R$ such that $f(e) = e - a \in \mathrm{U}(R)$. Therefore, a is strongly clean. $\square$

Based on the Cayley-Hamilton Theorem [48], we establish the following lemma.

Lemma 2.3.3 *Let R be a commutative ring and let $A \in \mathbb{M}_n(R)$. Let $f \in R[t]$ be a monic polynomial for which $f(A) = 0$. If $f(e)$ is a unit for some idempotent $e \in R$, then A is strongly clean.*

Proof Let e be such an idempotent. We show that $A - eI$ is a unit. In fact, using long division, we write $f(t) = (t - e)g(t) + f(e)$. Then $0 = f(A) = (A - eI)g(A) + f(e)I$. This gives that $(eI - A)g(A)f(e)^{-1} = I$, so $A - eI$ is invertible. Since eI is a central idempotent of $\mathbb{M}_n(R)$, we conclude that $A = (A - eI) + eI$ is strongly clean. $\square$

Corollary 2.3.4 *Let R be a commutative ring and let $A \in \mathbb{M}_n(R)$. If the characteristic polynomial χ_A of A has an n-SRC factorization, then A is strongly clean.*

Proof By hypothesis, there exist monic polynomials $f_0, f_1 \in R[t]$ such that $\chi_A = f_0 f_1$ and (f_0, f_1) is unimodular, and idempotents e_0, e_1 for which $f_0(e_0), f_1(e_1)$ are units. Find g_0, g_1 such that $f_0 g_0 + f_1 g_1 = 1$. By [8, Lemma 11], $\ker(f_0(A)) \oplus \ker(f_1(A)) = R^n$. It is clear that both $\ker(f_0(A))$ and $\ker(f_1(A))$ are A-invariant. Now, $A|_{\ker(f_0(A))}$ satisfies the polynomial f_0 and $A|_{\ker(f_1(A))}$ satisfies the polynomial f_1. By Lemma 2.3.3, $A|_{\ker(f_0(A))}$ and $A|_{\ker(f_1(A))}$ are strongly clean. It follows from [58] that A is strongly clean. Indeed, let $\varphi \in \mathrm{End}_R(R^n)$ be the projection of R^n onto $\ker(f_0(A))$, relative to the direct sum $R^n = \ker(f_0(A)) \oplus \ker(f_1(A))$. Then, $A\varphi = \varphi A$ and φA and $(1 - \varphi)A$ are strongly clean in $\varphi \mathbb{M}_n(R)\varphi$ and $(1 - \varphi)\mathbb{M}_n(R)(1 - \varphi)$, respectively. $\square$

Theorem 2.3.5 *If R is an n-SRC ring, then $\mathbb{M}_n(R)$ is strongly clean.*

Proof For any $A \in \mathbb{M}_n(R)$, the characteristic polynomial $\chi_A(t)$ of A has an n-SRC factorization. Thus A is strongly clean by Corollary 2.3.4. So $\mathbb{M}_n(R)$ is strongly clean. $\square$

It is worth making some comments on Theorem 2.3.5 and Definition 2.3.1.

Remark 2.3.6 (i) *Being an n-SRC ring is not necessary for the matrix ring $\mathbb{M}_n(R)$ to be strongly clean (see Example 2.3.15).*

(ii) *In Definition 2.3.1, we require $e_0 \neq e_1$. Allowing the idempotents to agree does not really gain anything, since given an n-SRC factorization $f = f_0 f_1$ with $e_0 = e_1$ and $f(e_0) \in \mathrm{U}(R)$, $f = f \cdot 1$ is an n-SRC factorization with respect to e_0 and any other idempotent.*

(iii) *Logically, allowing idempotents other than 0 and 1 to appear in Definition 2.3.1 is not as much of a generalization as we might think. But it can simplify computation. According to [58, Proposition 2], we have that if $\{e_1, e_2, \cdots, e_n\}$ is a set of complete orthogonal central idempotents, then $R = \oplus_{i=1}^{n} e_i R = \oplus_{i=1}^{n} e_i R e_i$, and R is strongly clean iff $e_i R e_i$ is strongly clean for $i = 1, \cdots, n$. Observe that, for any idempotent $e \in R$ (with R commutative) and $g(t) \in R[t]$, $g(e) = eg(1) + (1-e)g(0)$, and moreover, that $eg(1) = eg(e)$. In particular, $g(e)$ is a unit in R iff $eg(1) = eg(e)$ is a unit in the corner ring eR and $(1-e)g(0) = ((1-e)g)(0)$ is a unit in the corner ring $(1-e)R$. Thus, allowing two idempotents e_0 and e_1 for the polynomials $f_0(t)$ and $f_1(t)$ in an SR factorization $f = f_0 f_1$, we look at the associated four term direct sum decomposition corresponding to $e_0e_1 + e_0(1-e_1) + (1-e_0)e_1 + (1-e_0)(1-e_1) = 1$. Furthermore, we get a sum of f: $f = f_0 f_1 = e_0e_1 f_0 f_1 + e_0(1-e_1) f_0 f_1 + (1-e_0)e_1 f_0 f_1 + (1-e_0)(1-e_1) f_0 f_1$. $e_0e_1 f_0(t)$ and $e_0e_1 f_1(t)$ are units at the identity of e_0e_1R. $(1-e_0)(1-e_1)f_0(t)$ and $(1-e_0)(1-e_1)f_1(t)$ are units at 0 of $(1-e_0)(1-e_1)R$. In the other two factors, one of f_0 and f_1 (multiplied with corresponding identity of the corner rings) is a unit at the corresponding identity and the other is a unit at 0. So each component of $f_0 f_1$*

has an SR *factorization corresponding to the trivial idempotents* 0 *and* "1" *of the corresponding corner rings.*

(iv) *We still call the factorization an* SR *(*SRC*) factorization as in [8] because Definition 2.3.1 is essentially the same as that in [8] when we deal with the strong cleanness of matrix rings* $\mathbb{M}_n(R)$ *with* $n \geq 2$ *(see Theorem 2.3.14 below) although Definition 2.3.1 is really a generalization as Proposition 2.3.8 shows.*

Proposition 2.3.7 *Let* R *be an* n-SR *ring for some* $n \geq 4$*. Then* R *is local.*

Proof Suppose $e \in R$ is a nontrivial idempotent. Thus, R is a nontrivial direct product, say $R = R_1 \times R_2$, of rings. Consider the monic polynomial

$$h(t) = (t^{n-1}(t-1), t^{n-2}(t-1)^2) \in R[t] = R_1[t] \times R_2[t].$$

Suppose $h = fg$ is an n-SR factorization. Write $f = (f_1, f_2)$ and $g = (g_1, g_2)$. Clearly, f_1g_1 is an n-SR factorization of $t^{n-1}(t-1)$ in $R_1[t]$ and f_2g_2 is an n-SR factorization of $t^{n-2}(t-1)^2$ in $R_2[t]$.

Now, more generally, suppose fg is an n-SR factorization of $t^k(t-1)^{n-k}$ over an arbitrary non-zero commutative ring R. The same is then true via passing to a quotient $F = R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal. But F is a field, so $F[t]$ is a UFD, and it follows that the image of the monic polynomial f (resp., g) must be $t^i(t-1)^j$ for some i and j. If $ij \neq 0$, then $t^i(t-1)^j$ annihilates every idempotent, so in order for fg to be an n-SR factorization, f and g must be, in some order, t^i and $(t-1)^j$.

Returning to our previous situation, f_1 must have degree either $n-1$ or 1, whereas f_2 must have degree either $n-2$ or 2. Since $1 \neq 2$, $1 \neq n-2$, $2 \neq n-1$, and $n-1 \neq n-2$ for $n \geq 4$, we conclude that f cannot be monic, and hence h has no n-SR factorization. We conclude that every idempotent of R is trivial.

It remains to show that R is local. Observe that the n-SR property passes to quotient rings. In particular, if R has n-SR property, where $n \geq 4$, every quotient of R also has no nontrivial idempotents. Thus, suppose R has two distinct maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$.

By the Chinese Remainder Theorem [1, p.103], $R/(\mathfrak{m}_1 \cap \mathfrak{m}_2) \cong R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$, which clearly has nontrivial idempotents. We conclude that $\mathfrak{m}_1 = \mathfrak{m}_2$. It follows that R has a unique maximal ideal, so R is local, as desired. $\square$

Proposition 2.3.8 $\mathbb{C} \times \mathbb{C}$ *is an* n-SR *ring for* $n = 2, 3$.

Proof We assume $f(t) = f_0(t)f_1(t)$ with the fist factor as f_0 and the second as f_1.

(i) We prove that $\mathbb{C} \times \mathbb{C}$ is a 2-SR ring. Let

$$f(t) = (t^2 + a_1t + b_1, t^2 + a_2t + b_2) \in (\mathbb{C} \times \mathbb{C})[t] = \mathbb{C}[t] \times \mathbb{C}[t],$$

$a_1, b_1, a_2, b_2 \in \mathbb{C}$, be a monic polynomial of degree 2.

Case 1. $b_1 \neq 0, b_2 \neq 0$.

Then $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t + b_2) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$.

Case 2. $b_1 = 0, b_2 = 0$.

Then $f(t) = (t^2 + a_1t, t^2 + a_2t)$.

Subcase 1. $a_1 \neq 0, a_2 \neq 0$.

Then $f(t) = (t + a_1, t + a_2) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$.

Subcase 2. $a_1 = 0, a_2 = 0$.

Then $f(t) = (t^2, t^2) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (1, 1), e_1 = (0, 0)$.

Subcase 3. $a_1 \neq 0, a_2 = 0$.

Then $f(t) = (t + a_1, t) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

Subcase 4. $a_1 = 0, a_2 \neq 0$.

Similar to Subcase 3.

Case 3. $b_1 \neq 0, b_2 = 0$.

Then $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t)$.

Subcase 1. $a_1 \neq 0, a_2 \neq 0$.

Then $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t)$.

If $a_2 \neq -1$, then $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

If $a_2 = -1$, then

$$f(t) = (t^2 + a_1t + b_1, t^2 - t) = (t - t_1, t - 1) \cdot (t - t_2, t)$$

where $t_1 + t_2 = -a_1, t_1t_2 = b_1$ is an SR factorization with $e_0 = (0, 0), e_1 = (0, 1)$.

Subcase 2. $a_1 = 0, a_2 = 0$.

Then $f(t) = (t^2+b_1, t^2)\cdot(1, 1)$ is a trivial SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

Subcase 3. $a_1 \neq 0, a_2 = 0$.

Then $f(t) = (t^2+a_1t+b_1, t^2)\cdot(1, 1)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

Subcase 4. $a_1 = 0, a_2 \neq 0$.

Then $f(t) = (t^2 + b_1, t^2 + a_2t)$.

If $a_2 \neq -1$, then $f(t) = (t^2 + b_1, t^2 + a_2t) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

If $a_2 = -1$, then $f(t) = (t - t_1, t - 1) \cdot (t - t_2, t)$ where $t_1 + t_2 = 0, t_1t_2 = b_1$ is an SR factorization with $e_0 = (0, 0), e_1 = (0, 1)$.

Case 4. $b_1 = 0, b_2 \neq 0$.

Similar to Case 3.

(ii) We prove that $\mathbb{C} \times \mathbb{C}$ is a 3-SR ring. Let

$$f(t) = (t^3 + a_1t^2 + b_1t + c_1, t^3 + a_2t^2 + b_2t + c_2),$$

$a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{C}$, be a monic polynomial of degree 3.

Case 1. $c_1 \neq 0, c_2 \neq 0$.

Then $f(t) = (t^3 + a_1t^2 + b_1t + c_1, t^3 + a_2t^2 + b_2t + c_2) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$.

Case 2. $c_1 = 0, c_2 = 0$.

Then $f(t) = (t^3 + a_1t^2 + b_1t, t^3 + a_2t^2 + b_2t)$.

Subcase 1. $b_1 \neq 0, b_2 \neq 0$.

Then $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t + b_2) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$.

Subcase 2. $b_1 = 0, b_2 = 0$.

Then $f(t) = (t^3 + a_1t^2, t^3 + a_2t^2)$.

If $a_1 \neq 0, a_2 \neq 0$, then $f(t) = (t + a_1, t + a_2) \cdot (t^2, t^2)$ is a trivial SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$.

If $a_1 = a_2 = 0$, then $f(t) = (t^3, t^3) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (1, 1), e_1 = (0, 0)$.

If $a_1 = 0, a_2 \neq 0$, then $f(t) = (t^3, t^3 + a_2t^2) = (t, t + a_2) \cdot (t^2, t^2)$ is an SR factorization with $e_0 = (1, 0), e_1 = (1, 1)$.

If $a_1 \neq 0, a_2 = 0$, it is similar to the case $a_1 = 0, a_2 \neq 0$.

Subcase 3. $b_1 \neq 0, b_2 = 0$.

Then $f(t) = (t^3 + a_1t^2 + b_1t, t^3 + a_2t^2)$.

If $a_1 \neq 0, a_2 \neq 0$, then in case $a_2 \neq -1$, $f(t) = (t^2 + a_1t + b_1, t^2 + a_2t) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$, in case $a_2 = -1$, $f(t) = ((t - t_1)(t - t_2)t, (t - 1)t^2)$ where $t_1 + t_2 = -a_1, t_1t_2 = b_1$, when $t_1 \neq 1$ or $t_2 \neq 1$, say, $t_2 \neq 1$, $f(t) = (t - t_1, t - 1) \cdot (t^2 - t_2t, t^2)$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 1)$, when $t_1 = t_2 = 1$, $f(t) = ((t - 1)^2, t^2) \cdot (t, t - 1)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 0)$.

If $a_1 = a_2 = 0$, then $f(t) = (t^3 + b_1t, t^3) = (t^2 + b_1, t^2) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

If $a_1 = 0, a_2 \neq 0$, then in case $a_2 \neq -1$, $f(t) = (t^2 + b_1, t^2 + a_2t) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$, in case $a_2 = -1$, $f(t) = (t^2+b_1, t^2)\cdot(t, t-1)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 0)$.

If $a_1 \neq 0, a_2 = 0$, then $f(t) = (t^2 + a_1t + b_1, t^2) \cdot (t, t)$ is an SR factorization with $e_0 = (0, 1), e_1 = (1, 1)$.

Subcase 4. $b_1 = 0, b_2 \neq 0$.

Similar to Subcase 3.

Case 3. $c_1 = 0, c_2 \neq 0$.

Then $f(t) = (t^3 + a_1t^2 + b_1t, t^3 + a_2t^2 + b_2t + c_2)$.

Subcase 1. $b_1 \neq 0, b_2 \neq 0$.

Then $f(t) = (t^2 + a_1t + b_1, (t - t_1)(t - t_2)) \cdot (t, t - t_3)$ where $t_1 + t_2 + t_3 = -a_2$, $t_1t_2 + t_1t_3 + t_2t_3 = b_2$, and $t_1t_2t_3 = -c_2$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 0)$.

Subcase 2. $b_1 = 0, b_2 = 0$.

Then $f(t) = (t^3 + a_1t^2, t^3 + a_2t^2 + c_2)$.

If $a_1 \neq 0, a_2 \neq 0$, then $f(t) = (t+a_1, t-t_1)\cdot(t^2, (t-t_2)(t-t_3))$ where $t_1+t_2+t_3 = -a_2$, $t_1t_2t_3 = -c_2$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 0)$.

If $a_1 = a_2 = 0$, then $f(t) = (t^3, t^3 + c_2) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (1, 0), e_1 = (1, 1)$.

If $a_1 = 0, a_2 \neq 0$, then $f(t) = (t^3, t^3 + a_2t^2 + c_2) \cdot (1, 1)$ is a trivial SR factorization with $e_0 = (1, 0), e_1 = (1, 1)$.

If $a_1 \neq 0, a_2 = 0$, then $f(t) = (t + a_1, t - t_1) \cdot (t^2, (t - t_2)(t - t_3))$ where $t_1 + t_2 + t_3 = 0$, $t_1t_2t_3 = -c_2$ is an SR factorization with $e_0 = (0, 0), e_1 = (1, 0)$.

Subcase 3. $b_1 \neq 0, b_2 = 0$.

Then

$$f(t) = (t^3 + a_1t^2 + b_1t, t^3 + a_2t^2 + c_2) = (t^2 + a_1t + b_1, (t-t_1)(t-t_2)) \cdot (t, t-t_3)$$

with $t_1+t_2+t_3 = -a_2, t_1t_2t_3 = -c_2$ is an SR factorization with $e_0 = (0,0), e_1 = (1,0)$.

Subcase 4. $b_1 = 0, b_2 \neq 0$.

Then $f(t) = (t^3 + a_1t^2, t^3 + a_2t^2 + b_2t + c_2)$.

If $a_1 \neq -1$, then $f(t) = (t^3+a_1t^2, t^3+a_2t^2+b_2t+c_2)\cdot(1,1)$ is a trivial SR factorization with $e_0 = (1,0), e_1 = (1,1)$.

If $a_1 = -1$, then $f(t) = (t^3-t^2, t^3+a_2t^2+b_2t+c_2) = (t-1, t-t_1)\cdot(t^2, (t-t_2)(t-t_3))$ with $t_1+t_2+t_3 = -a_2, t_1t_2t_3 = -c_2$ is an SR factorization with $e_0 = (0,0), e_1 = (1,0)$.

Case 4. $c_1 \neq 0, c_2 = 0$.

Similar to Case 3.

□

Remark 2.3.9 (i) *By Proposition 2.3.8, we know that the hypothesis $n \geq 4$ in Proposition 2.3.7 is necessary.*

(ii) *$\mathbb{C} \times \mathbb{C}$ is not an n-SRC ring for $n \geq 2$ by the fact that $\mathbb{C} \times \mathbb{C}$ is not local and Proposition 2.3.13 below.*

(iii) *In Proposition 2.3.8, $\mathbb{C}$ can be replaced by any algebraically closed field.*

Proposition 2.3.10 *Let $n = 2, 3$ and let R be an n-SR ring. If $R[t]$ has an irreducible monic polynomial of degree n, then R has only the trivial idempotents.*

Proof Let $f \in R[t]$ be irreducible, monic, and of degree n. Suppose $e \in R$ is a non-trivial idempotent. Then R can be regarded as the direct product of eR and $(1-e)R$.

It follows that either $ef(t)$ or $(1-e)f(t)$ is an irreducible polynomial, since otherwise, both factorizations must be into monic polynomials of degrees 1 and $n-1$, respectively, and then we can factor f as a product of two monic polynomials of degrees 1 and $n-1$, respectively. Without loss of generality, suppose $g(t) = ef(t)$ is irreducible in $eR[t]$. Consider the monic polynomial $f' = (g(t), t^{n-1}(t-1)) \in R[t]$. Any n-SR factorization of f' must have first coordinate degree either 0 or n since g is irreducible. On the other hand, the second coordinate, as in the proof of Proposition 2.3.7, must have degree 1 or $n-1$, and it follows as in that proof, since $0 \neq 1$, $0 \neq n-1$, $1 \neq n$, and $n-1 \neq n$ for $n = 2$ or 3, that f' has no n-SR factorization. We conclude from this contradiction that R has no nontrivial idempotents. $\square$

Definition 2.3.11 *[21, p.17] A ring R is called* **projective-free** *if every finitely generated projective R-module is free of unique rank.*

Camillo and Yu [13] proved that R is semiperfect iff R is clean and **I-finite** (i.e. R does not have an infinite set of non-zero orthogonal idempotents). For a projective-free ring, we have the following result.

Proposition 2.3.12 *Let R be a projective-free ring. Then the following are equivalent:*

(i) *R is a strongly clean ring.*

(ii) *R is a clean ring.*

(iii) *R is a local ring.*

(iv) *R is an exchange ring.*

(v) *R is a semiperfect ring.*

If, in addition, R is commutative, then the foregoing are equivalent to the following:

(vi) *R is a 1-SR ring.*

(vii) *R is a* 1-SRC *ring.*

Proof (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (iv) This is a well-known result in [57].

(iv) $\Rightarrow$ (iii) We prove that R has only 0 and 1 as its idempotents. Suppose $e^2 = e \in R$. Then $R = Re \oplus R(1-e)$. Since R is projective-free, we get $Re = 0$ or $R(1-e) = 0$. So $e = 0$ or $e = 1$. Let $r \notin \mathrm{U}(R)$. Then, because R is an exchange ring, there exists $e^2 = e$ such that $e \in Rr$ and $1 - e \in R(1-r)$. That is, $1 \in Rr$ or $1 \in R(1-r)$. But $r \notin \mathrm{U}(R)$, so $1 \in R(1-r)$. Similarly, $1 \in (1-r)R$. So $1 - r \in \mathrm{U}(R)$. Therefore, R is local.

(iii) $\Rightarrow$ (v) This is evident.

(v) $\Rightarrow$ (ii) This is a result of [13].

(i) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) This is Theorem 2.3.2. □

We have not determined whether the rings in Proposition 2.3.10 must be local under the hypothesis. However, the SRC hypothesis forces locality for $n \geq 2$, as the next proposition shows.

Proposition 2.3.13 *Let R be an n-*SRC *ring for some $n \geq 2$. Then R is local.*

Proof By Theorem 2.3.5, $\mathbb{M}_n(R)$ is strongly clean. Since it is known that strong cleanness passes to corners, R must be a strongly clean ring. It suffices to show that R has no nontrivial idempotents since a ring with only trivial idempotents is strongly clean iff it is local. The result follows from Proposition 2.3.7 for $n \geq 4$. However, we give a differently elementary argument that works for all $n \geq 2$. Suppose $e \in R$ is a nontrivial idempotent. Consider the polynomial $f(t) = t^n - et \in R[t]$. Since R is an n-SRC ring, there is a factorization $f(t) = f_0(t)f_1(t)$ of $f(t)$ into monic polynomials such that $(f_0(t), f_1(t))$ is unimodular and there are idempotents $e_0, e_1 \in R$ such that $f_0(e_0), f_1(e_1)$ are units in R. We verify that such a factorization cannot exist.

A trivial factorization cannot occur since if $g^2 = g \in R$ then $f(g) = g(1-e)$ cannot be a unit. Thus, f_0, f_1 are unimodular polynomials, and each has degree at least 1. It is enough to show that f does not have a nontrivial factorization as a product of a pair of unimodular monic polynomials. Indeed, let $f = f_0 f_1$ be such a factorization. Since e is not a unit, $e \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Since the images of f_0 and f_1 are unimodular in $(R/\mathfrak{m})[t]$, we may assume that R is a field and that $f(t) = t^n$. But $R[t]$ is then a UFD, in which f_0 and f_1, must be, up to units, each power of t, but this forces $f_0 R[t] + f_1 R[t] \subseteq tR[t]$ since f_0 and f_1 were monic polynomials with degree at least 1. This is a contradiction, and we conclude that the original strongly clean ring R has only the trivial idempotents, and hence is local. □

Now we immediately get the following result.

Theorem 2.3.14 *Let R be a commutative ring and $n \geq 2$. Then the following are equivalent:*

(i) *R is an n-SRC ring .*

(ii) *R is a local n-SRC ring.*

(iii) *R is local and $\mathbb{M}_n(R)$ is strongly clean.*

Proof (i) $\Leftrightarrow$ (ii) By Proposition 2.3.13.

(ii) $\Rightarrow$ (iii) By Theorem 2.3.5.

(iii) $\Rightarrow$ (ii) By [8, Corollary 15]. □

But for a commutative ring R, being an n-SRC ring is not a necessary condition for $\mathbb{M}_n(R)$ to be strongly clean.

Example 2.3.15 *Let R be a Boolean ring with more than 2 elements. Then R is not an n-SRC ring for $n \geq 2$ by Proposition 2.3.13. However, $\mathbb{M}_n(R)$ is strongly clean for any $n \in \mathbb{N}$ [78, Corollary 3.2].*

2.4 Strong cleanness of matrices over projective-free rings or rings having ULP

Section 2.3 shows that the theory of SRC factorization can not provide us with new classes of strongly clean matrix rings except the known local ones. However, it can help us to find all strongly clean matrices over commutative projective-free rings or commutative rings having ULP (see Definition 2.4.5) even though the matrix ring is not strongly clean.

Definition 2.4.1 (i) *A matrix $A \in \mathbb{M}_n(R)$ is called **singular** if A is non-invertible and **nonsingular** if A is invertible.*

(ii) *A singular matrix $A \in \mathbb{M}_n(R)$ is called **purely singular** if $I - A$ is singular or **semi-purely singular** if $I - A$ is nonsingular.*

(iii) *A nonsingular matrix $A \in \mathbb{M}_n(R)$ is called **purely nonsingular** if $I - A$ is nonsingular or **semi-purely nonsingular** if $I - A$ is singular.*

Every matrix belongs to exactly one of the above four types. All types of matrices are strongly clean except purely singular ones. So we have the following lemma.

Lemma 2.4.2 *The matrix ring $\mathbb{M}_n(R)$ is strongly clean iff its purely singular matrices are strongly clean.*

Lemma 2.4.3 *[79, Lemma 2] Let R be a projective-free ring. Then a purely singular matrix $T \in \mathbb{M}_n(R)$ is strongly clean iff T is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular.*

By this lemma, we get a necessary condition for a matrix to be strongly clean when R is commutative projective-free.

Corollary 2.4.4 *Let R be a commutative projective-free ring. If $T \in \mathbb{M}_n(R)$ is strongly clean, then $\chi_T(t)$ has an n-SR factorization.*

Proof If T is nonsingular, then $\chi_T(t) = \det(tI - T) = f_0(t)f_1(t) = \chi_T(t) \cdot 1$ with $f_0(t) = \chi_T(t)$, $f_1(t) = 1$, $e_0 = 0$, and $e_1 = 1$ is an n-SR factorization. If T is semi-purely singular, then $\chi_T(t) = \det(tI - T) = f_0(t)f_1(t) = 1 \cdot \chi_T(t)$ with $f_0(t) = 1$, $f_1(t) = \chi_T(t)$, $e_0 = 0$, and $e_1 = 1$ is an n-SR factorization. If T is purely singular, then by Lemma 2.4.3, T is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular. So $\chi_T(t) = \chi_{T_0}(t) \cdot \chi_{T_1}(t)$ with $f_0(t) = \chi_{T_0}(t)$, $f_1(t) = \chi_{T_1}(t)$, $e_0 = 0$, and $e_1 = 1$ is an n-SR factorization. $\square$

Definition 2.4.5 *A commutative ring R is said to have the* **unimodular lifting property (ULP** *for short) if, for any pair $(f_0(t), f_1(t))$ of monic polynomials in $R[t]$, the unimodularity of $(\overline{f}_0(t), \overline{f}_1(t))$ in $\frac{R}{\mathfrak{m}}[t]$ for all $\mathfrak{m} \in \mathrm{Max}(R)$ implies the unimodularity of $(f_0(t), f_1(t))$ in $R[t]$.*

The class of commutative projective-free rings having ULP includes commutative local rings [43, Example 1.6] [44, Theorem 19.29], PID [63, p.177, Theorem 1], and polynomial rings with finitely many indeterminates over a field (Quillen-Suslin Theorem [65]).

Proposition 2.4.6 *Each commutative semilocal ring has* ULP.

Proof Let R be a commutative semilocal ring. Then R has finitely many maximal ideals, say $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$. Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $(\overline{f}_0(t), \overline{f}_1(t))$ be unimodular in $\frac{R}{\mathfrak{m}_k}[t]$ for $k = 1, 2, \cdots, n$. Since $\overline{f}_0(t)\frac{R}{\mathfrak{m}_k}[t] + \overline{f}_1(t)\frac{R}{\mathfrak{m}_k}[t] = \frac{R}{\mathfrak{m}_k}[t]$, we get $f_0(t)R[t] + f_1(t)R[t] + \mathfrak{m}_k[t] = R[t]$. Hence, $f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t) = 1$ for some $a_k(t), b_k(t) \in R[t]$, and $c_k(t) \in \mathfrak{m}_k[t]$. Therefore,

$$1 = \Pi_{k=1}^{n} \left(f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t)\right) = f_0(t)a'(t) + f_1(t)b'(t) + c'(t)$$

for some $a'(t), b'(t) \in R[t]$, and $c'(t) \in J(R)[t]$. Thus,

$$R[t] = f_0(t)R[t] + f_1(t)R[t] + c'(t)R[t] = f_0(t)R[t] + f_1(t)R[t] + J(R)R[t].$$

Notice that $\frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]}$ is a finitely generated R-module and

$$J(R)\frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]} = \frac{J(R)R[t]+f_0(t)R[t]+f_1(t)R[t]}{f_0(t)R[t]+f_1(t)R[t]} = \frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]}.$$

So $f_0(t)R[t]+f_1(t)R[t] = R[t]$ by the Nakayama's Lemma [1, Corollary 15.13]. Therefore, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. □

Corollary 2.4.7 *Commutative local rings have* ULP.

Proposition 2.4.8 *Every* UFD *has* ULP.

Proof Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $\left(\overline{f_0}(t), \overline{f_1}(t)\right)$ be unimodular in $\frac{R}{\mathfrak{m}}[t]$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = 1$ in $\frac{R}{\mathfrak{m}}[t]$. We want to prove that $\gcd(f_0(t), f_1(t))$ is a unit in $R[t]$. Suppose $\gcd(f_0(t), f_1(t))$ is not a unit.

Case 1. $\gcd(f_0(t), f_1(t)) = m \in R$ with $m \notin \mathrm{U}(R)$.

Then there exists $\mathfrak{m}_o \in \mathrm{Max}(R)$ such that $m \in \mathfrak{m}_o$. So $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = \overline{m} = 0$ in $\frac{R}{\mathfrak{m}_o}[t]$. This is a contradiction.

Case 2. $\gcd(f_0(t), f_1(t)) = g(t) \in R[t]$ with $\deg(g(t)) \geq 1$ in $R[t]$.

Then, for any $\mathfrak{m} \in \mathrm{Max}(R)$, $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) \neq 1$ in $\frac{R}{\mathfrak{m}}[t]$ because the coefficient of the leading term of $g(t)$ is a unit.

Hence, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. □

Given a monic polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in R[t]$, the matrix

$$C_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

is called the **companion matrix** of $f(t)$.

Lemma 2.4.9 *[41, Theorem VII.4.3] Let F be a field and $f(t)$ be a monic polynomial in $F[t]$. Then $f(t)$ is the characteristic and minimal polynomial of the companion matrix C_f.*

Theorem 2.4.10 *Let R be a commutative ring having* ULP *and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in R[t]$. Then the companion matrix C_f is strongly clean iff $\chi_{C_f}(t) = f(t)$ has an n-*SRC *factorization.*

Proof $\Leftarrow$ By Corollary 2.3.4.

$\Rightarrow$ The argument of Corollary 2.4.4 shows that if T is not purely singular, then $\chi_T(t)$ has a trivial SRC factorization. So we may assume that C_f is purely singular. Then by Lemma 2.4.3, there exists $P \in \mathbb{M}_n(R)$ such that $P^{-1}C_fP = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ with T_0 being $k \times k$ semi-purely nonsingular matrix and T_1 being $(n-k) \times (n-k)$ semi-purely singular matrix where $0 < k < n$. Then, for every maximal ideal $\mathfrak{m}$ in R, $\overline{C_f} = C_{\overline{f}} \in \mathbb{M}_n(\frac{R}{\mathfrak{m}})$ has $\overline{f}(t) \in \frac{R}{\mathfrak{m}}[t]$ as the characteristic and minimal polynomial by Lemma 2.4.9. So

$$\overline{f}(t) = \chi_{\overline{C_f}}(t) = \chi_{\overline{T}_0}(t) \cdot \chi_{\overline{T}_1}(t) = \det(tI_k - \overline{T}_0) \cdot \det(tI_{n-k} - \overline{T}_1).$$

If $\gcd(\det(tI_k - \overline{T}_0), \det(tI_{n-k} - \overline{T}_1)) = g(t)$ with $\deg(g(t)) \geq 1$, then the minimal polynomial of $\overline{C_f}$ is $\frac{\det(tI_k - \overline{T}_0) \cdot \det(tI_{n-k} - \overline{T}_1)}{g(t)}$ which has degree less than $\deg(\chi_{\overline{C_f}}) = \deg(f)$. This is a contradiction. So $f_0(t) = \det(tI - T_0)$, $f_1(t) = \det(tI - T_1)$, $e_i = i$, and $f_i(e_i) \in \mathrm{U}(R)$ $(i = 0, 1)$ yield an n-SRC factorization of $\chi_{C_f}(t) = f(t)$. $\square$

Corollary 2.4.11 *Let R be a commutative ring having* ULP *and let $f(t) \in R[t]$ be a monic polynomial with* $\deg(f(t)) = n$. *Then the following are equivalent:*

(i) *For all $A \in \mathbb{M}_n(R)$ with $\chi_A(t) = f(t)$, A is strongly clean.*

(ii) *The companion matrix C_f is strongly clean.*

(iii) *$f(t)$ has an n-*SRC *factorization.*

Proof (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is Theorem 2.4.10. (iii) $\Rightarrow$ (i) is Corollary 2.3.4. $\square$

Theorem 2.4.12 *Let R be a commutative projective-free ring. Then a purely singular matrix $A \in \mathbb{M}_n(R)$ is strongly clean iff $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ with $e_i = i$ $(i = 0, 1)$ and A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$.*

Proof $\Rightarrow$ By Lemma 2.4.3, A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular. By Corollary 2.4.4, $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ where $\chi_{T_0}(t) = f_0(t)$, $\chi_{T_1}(t) = f_1(t)$, $e_i = i$, $f_i(e_i) \in \mathrm{U}(R)$ $(i = 0, 1)$.

$\Leftarrow$ By Corollary 2.4.11, T_0 and T_1 are strongly clean because $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$ have trivial SRC factorizations. Consequently, A is strongly clean because the strongly clean property is invariant under similarity. $\square$

2.5 Strong cleanness of $\mathbb{M}_n(\mathrm{C}(X))$ and $\mathbb{M}_n(\mathrm{C}(X, \mathbb{C}))$

A completely regular space X is called a **P-space** (**P-space relative to** $\mathbb{C}$) if every prime ideal in $\mathrm{C}(X)$ $(\mathrm{C}(X, \mathbb{C}))$ is maximal. It is well known that every discrete space is a P-space and every P-space is strongly zero-dimensional. In this section, we prove that matrix rings over $\mathrm{C}(X)$ $(\mathrm{C}(X, \mathbb{C}))$ with X a Hausdorff P-space (P-space relative to $\mathbb{C}$) are strongly π-regular (hence, strongly clean). To this end, we need some notions. An ideal $I \leq \mathrm{C}(X, \mathbb{C})$ is a ***z*-ideal** if $z(g) \in z[I] = \{z(f) : f \in I\}$ implies $g \in I$. Let S be a ring and R be a subring of S such that they share the same identity. The ring S is called a **finite extension** of R if S, as an R-module, is generated by a finite subset $X \subseteq S$ of generators.

Theorem 2.5.1 *Let X be a* P*-space. Then every finite extension of* $\mathrm{C}(X)$ *is strongly π-regular. In particular,* $\mathbb{M}_n(\mathrm{C}(X))$ *is strongly π-regular.*

Proof Let X be a P-space. Then $\mathrm{C}(X)$ is a regular ring by [31, 4J], so $\mathrm{C}(X)$ is π-regular. By [40, Theorem 2, Corollary 4], every finite extension of $\mathrm{C}(X)$ is strongly π-regular. $\mathbb{M}_n(\mathrm{C}(X))$ is the finite extension of $\mathrm{C}(X)$ with generator set $\{E_{ij} : i, j = 1, \cdots, n\}$ where E_{ij} is the matrix with the (i, j)-entry being 1 and other entries 0. Hence, $\mathbb{M}_n(\mathrm{C}(X))$ is strongly π-regular. $\square$

Corollary 2.5.2 *Let X be a* P*-space and G a locally finite group. Then* $\mathbb{M}_n((\mathrm{C}(X)G)[[X]])$ *and* $\mathbb{M}_n\left(\frac{(\mathrm{C}(X)G)[x]}{(x^k)}\right)$ *are strongly clean. In particular,* $\mathbb{M}_n(\mathrm{C}(X))$ *is strongly clean.*

Proof By Theorem 2.5.1 and [78, Corollary 3.2]. $\square$

Corollary 2.5.3 *If X is a discrete space, then* $\mathbb{M}_n(\mathrm{C}(X))$ *is strongly π-regular and hence strongly clean.*

Proof Every discrete space is a P-space. $\square$

Theorem 2.5.4 *Let X be a Hausdorff* P*-space relative to* $\mathbb{C}$*. Then* $R = \mathrm{C}(X, \mathbb{C})$ *is a regular ring. Hence every finite extension of R is strongly π-regular. In particular,* $\mathbb{M}_n(R)$ *is strongly π-regular.*

Proof Suppose X is a P-space relative to $\mathbb{C}$. For $p \in X$, set $O_p = \{f \in R : z(f)$ is a neighborhood of $p\}$ and $M_p = \{f \in R : f(p) = 0\}$. Then M_p is a maximal ideal and O_p is a z-ideal in R with $O_p \subseteq M_p$.

Let A_p be the family of all zero-sets containing a given point p. Then A_p is the unique z-ultrafilter converging to p [31, p.47]. For any ideal I in $R, z[I]$ is a z-filter, and if I is a maximal ideal then $z[I]$ is a z-ultrafilter. Thus $z[O_p] \subseteq z[M_p] = A_p$. So M_p is the only maximal ideal that contains O_p. Notice that $z(f^n) = z(f)$ for any $n \in \mathbb{N}$. If I is a z-ideal and $f^n \in I$, then $z(f) = z(f^n) \in z[I]$ implies $f \in I$. So I is a radical ideal, that is, I is an intersection of prime ideals containing I. Hence O_p is an intersection of prime ideals. Since M_p is the only maximal ideal that contains O_p, $O_p \neq M_p$ implies that O_p is contained in a prime ideal that is not maximal. However, every prime ideal is maximal if X is a P-space relative to $\mathbb{C}$. Thus, $O_p = M_p$.

Let p be any point in $z(f)$. Then $f(p) = 0$ implies $f \in M_p = O_p$. Hence $z(f)$ is open, that is, every zero-set is clopen. Suppose I is an ideal of R and $z(f) \in z[I]$. Then $z(f) = z(g)$ for some $g \in I$. Define $h : X \to \mathbb{C}$ by $h(x) = 0$ if $x \in z(f)$ and $h(x) = \frac{f(x)}{g(x)}$ if $x \notin z(f)$. Then $h \in R$ and $f = gh$. Thus, $f \in I$, so I is a z-ideal. Hence every ideal in R is a z-ideal. So every ideal is a radical ideal.

Since f and f^2 belong to the same prime ideals,

$$(f) = \cap S = \cap T = (f^2),$$

where $S = \{\mathfrak{p} \in \mathrm{Max}(R) : f \in \mathfrak{p}\}$ and $T = \{\mathfrak{p} \in \mathrm{Max}(R) : f^2 \in \mathfrak{p}\}$. So $f = f^2 f_0$ for some $f_0 \in R$. Therefore, R is a regular ring. Hence, by [40, Theorem 2, Corollary 4], every finite extension of R is strongly π-regular. In particular, $\mathbb{M}_n(R)$ is strongly π-regular since $\mathbb{M}_n(R)$ is the finite extension of R. $\square$

Remark 2.5.5 *Corollary 2.5.2 and Corollary 2.5.3 still hold for* $\mathrm{C}(X, \mathbb{C})$.

Chapter 3

$g(x)$-Clean and Strongly $g(x)$-Clean Rings

This chapter is divided into two parts.

In Section 3.1, we discuss the general properties of $g(x)$-clean rings which are similar to those of clean rings, completely determine the relations between $g(x)$-clean rings and clean rings, and handle the $(x^2 + cx + d)$-clean and $(x^n - x)$-clean rings. This section is adapted from [28].

In Section 3.2, we define strongly $g(x)$-clean rings and give some properties of strongly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings and strongly clean rings. Moreover, we establish the relations between strongly $g(x)$-clean and strongly clean rings.

3.1 $g(x)$-clean rings

From definitions it turns out that the $(x^2 - x)$-clean rings are precisely the clean rings. The following two examples explain the relations between $g(x)$-clean rings and clean rings.

Let $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : n$ and p are coprime with p prime$\}$ be the localization of $\mathbb{Z}$ at the prime ideal (p) and C_3 be the cyclic group of order 3.

Example 3.1.1 *There exists an $(x^4 - x)$-clean ring which is not clean. The proof of [80, Theorem 3.1] shows that $\mathbb{Z}_{(7)}C_3$ is an $(x^4 - x)$-clean ring. But $\mathbb{Z}_{(7)}C_3$ is not clean by [36, Example 1].*

Example 3.1.2 *Let R be a Boolean ring containing more than two elements. If $c \in R$ with $0 \neq c \neq 1$ and $g(x) = x^2 + (1+c)x + c = (x+1)(x+c)$, then R is not $g(x)$-clean. Actually, if $c = s + u$ where $g(s) = 0$ and $u \in \mathrm{U}(R)$, then $u = 1$ and so $s = c - 1 = c + 1$. But, clearly, $g(c+1) \neq 0$. However, R is certainly clean.*

Let R and S be rings and $\theta : \mathrm{Z}(R) \longrightarrow \mathrm{Z}(S)$ be a ring homomorphism with $\theta(1) = 1$. For $g(x) = \Sigma a_i x^i \in \mathrm{Z}(R)[x]$, let $\theta'(g(x)) = \Sigma\theta(a_i)x^i \in \mathrm{Z}(S)[x]$. Then θ induces a map θ' from $\mathrm{Z}(R)[x]$ to $\mathrm{Z}(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$. We give some properties of $g(x)$-clean rings which are similar to those of clean rings.

Proposition 3.1.3 *Let $\theta : R \longrightarrow S$ be a ring epimorphism. If R is $g(x)$-clean, then S is $\theta'(g(x))$-clean.*

Proof Let $g(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathrm{Z}(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in \mathrm{Z}(S)[x]$. For any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is $g(x)$-clean, there exist $s_0 \in R$ and $u \in \mathrm{U}(R)$ such that $r = s_0 + u$ and $g(s_0) = 0$. Then $s = \theta(r) = \theta(s_0) + \theta(u)$ with $\theta(u) \in \mathrm{U}(S)$ and $\theta'(g(\theta(s_0)) = 0$, that is, S is $\theta'(g(x))$-clean. $\square$

Corollary 3.1.4 *Let R be $g(x)$-clean. Then, for any ideal I of R, R/I is $\overline{g}(x)$-clean where $\overline{g}(x) \in \mathrm{Z}(R/I)[x]$.*

Proof This is because R/I is the homomorphic image of R. □

Proposition 3.1.5 *Let $I \leq J(R)$ be an ideal of R, $\eta : R \to R/I$ with $\eta(r) = r + I = \overline{r}$, and $g(x) = \sum_{i=0}^{n} a_i x^i \in Z(R)[x]$ with $\overline{g}(x) = \sum_{i=0}^{n} \overline{a}_i x^i \in Z(R/I)[x]$. If R/I is $\overline{g}(x)$-clean and roots of $\overline{g}(x)$ lift modulo I, then R is $g(x)$-clean .*

Proof For any $r \in R$, let $r + I = \overline{r} = \overline{s} + \overline{u}$ where $\overline{g}(\overline{s}) = 0$ and $\overline{u} \in U(R/I)$. Because roots of $\overline{g}(x)$ lift modulo I, we may assume that $s \in R$ satisfies $g(s) = 0$. So $r - s - u = i$ for some $i \in I$. Hence, $r = s + (u + i)$ with $u + i \in U(R)$. Thus, r is $g(x)$-clean, that is, R is $g(x)$-clean. □

Proposition 3.1.6 *Let $g(x) \in \mathbb{Z}[x]$ and let $\{R_i\}_{i \in I}$ be a family of rings. Then the direct product $\prod_{i \in I} R_i$ is $g(x)$-clean iff every R_i, $i \in I$, is $g(x)$-clean.*

Proof $\Rightarrow$ By Proposition 3.1.3.

$\Leftarrow$ Let $r = (r_i)_{i \in I} \in \prod_{i \in I} R_i$. Then $r_i = s_i + u_i$ with $g(s_i) = 0$ and $u_i \in U(R_i)$ for any $i \in I$ by $g(x)$-cleanness of R_i. Set $s = (s_i)_{i \in I}$ and $u = (u_i)_{i \in I}$. Then $r = s + u, g(s) = 0$, and $u \in U(\prod_{i \in I} R_i)$. Consequently, $\prod_{i \in I} R_i$ is $g(x)$-clean. □

Define $\pi_n : Z(R) \longrightarrow \mathbb{M}_n(R)$ by $\pi(a) = aI_n$ with I_n being the identity matrix of $\mathbb{M}_n(R)$. Clearly, $\mathbb{M}_n(R)$ is a $Z(R)$-algebra. Furthermore, we have the following result.

Proposition 3.1.7 *Let R be a ring, $g(x) \in Z(R)[x]$, and $n \in \mathbb{N}$. Then R is $g(x)$-clean iff the upper triangular matrix ring $\mathbb{T}_n(R)$ is $g(x)$-clean.*

Proof $\Rightarrow$ Let $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since R is $g(x)$-clean, for any $1 \leq i \leq n$, there exist $s_{ii} \in R$ and $u_{ii} \in U(R)$ such that $a_{ii} = s_{ii} + u_{ii}$ with $g(s_{ii}) = 0$. Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in Z(R)[x]$. Let $A = S + U$ with

$$S=\begin{pmatrix} s_{11} & 0 & \cdots & 0 \\ 0 & s_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{nn} \end{pmatrix} \text{ and } U=\begin{pmatrix} u_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & u_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{pmatrix}.$$

Then $U \in GL_n(R)$ and

$$\begin{aligned} g(S) &= a_0 I_n + a_1 S + \cdots + a_m S^m \\ &= \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} + \begin{pmatrix} a_1 s_{11} & 0 & \cdots & 0 \\ 0 & a_1 s_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 s_{nn} \end{pmatrix} + \cdots + \begin{pmatrix} a_m s_{11}^m & 0 & \cdots & 0 \\ 0 & a_m s_{22}^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m s_{nn}^m \end{pmatrix} \\ &= \begin{pmatrix} g(s_{11}) & 0 & \cdots & 0 \\ 0 & g(s_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(s_{nn}) \end{pmatrix} \\ &= 0. \end{aligned}$$

So $\mathbb{T}_n(R)$ is $g(x)$-clean.

$\Leftarrow$ For any $a \in R$, let A be the diagonal matrix $\text{diag}(a, \cdots, a)$. Then by a direct computation, $g(x)$-cleanness of A implies that $g(x)$-cleanness of a. $\square$

In [36], the authors proved that if R is clean then so is $\mathbb{M}_n(R)$ for any $n \in \mathbb{N}$. Here, we have a similar result for $g(x)$-clean rings.

Proposition 3.1.8 *Let R be a ring and $g(x) \in \mathrm{Z}(R)[x]$. If R is $g(x)$-clean, then $\mathbb{M}_n(R)$ is $g(x)$-clean for any $n \in \mathbb{N}$.*

Proof We prove the proposition by induction on n. The case $n = 1$ is clear. Assume that the proposition holds for $\mathbb{M}_{n-1}(R)$ where $n > 1$. If $\alpha \in \mathbb{M}_n(R)$, then $\alpha = \begin{pmatrix} A & X \\ Y & b \end{pmatrix}$ in block form where $A \in \mathbb{M}_{n-1}(R)$ and $b \in R$. By hypothesis, $A = S + U$ where $S \in \mathbb{M}_{n-1}(R)$ is a root of $g(x)$ and U is a unit of $\mathbb{M}_{n-1}(R)$. Then $b - YU^{-1}X \in R$. Since R is $g(x)$-clean, we have $b - YU^{-1}X = s + u$ where $s \in R$ is a root of $g(x)$ and $u \in \mathrm{U}(R)$.

Then

$$\alpha - \begin{pmatrix} S & 0 \\ 0 & s \end{pmatrix} = \beta, \text{ where } \beta = \begin{pmatrix} U & X \\ Y & u + YU^{-1}X \end{pmatrix}.$$

By computation, we have

$$\begin{pmatrix} I_{n-1} & 0 \\ -YU^{-1} & 1 \end{pmatrix} \begin{pmatrix} U & X \\ Y & u + YU^{-1}X \end{pmatrix} \begin{pmatrix} I_{n-1} & -U^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & u \end{pmatrix}.$$

So β is a unit of $\mathbb{M}_n(R)$. Since $\begin{pmatrix} S & 0 \\ 0 & s \end{pmatrix}$ is a root of $g(x)$, $\alpha \in \mathbb{M}_n(R)$ is $g(x)$-clean. □

Proposition 3.1.9 *Let R be a ring and $g(x) \in \mathrm{Z}(R)[x]$. Then the formal power series ring $R[[t]]$ is $g(x)$-clean iff R is $g(x)$-clean.*

Proof $\Leftarrow$ Let $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$. Since R is $g(x)$-clean, $a_0 = s + u$ where $g(s) = 0$ and $u \in \mathrm{U}(R)$. Then $f = s + (u + \sum_{i\geq 1} a_i t^i)$ with $u + \sum_{i\geq 1} a_i t^i \in \mathrm{U}(R[[t]])$. So f is $g(x)$-clean. Hence $R[[t]]$ is $g(x)$-clean.

$\Rightarrow$ Since $\theta : R[[t]] \to R$, sending $\sum_{i\geq 0} a_i t^i$ to a_0, is a ring epimorphism. By Proposition 3.1.3, R is $g(x)$-clean. □

Remark 3.1.10 *Generally speaking, the polynomial ring $R[t]$ is not $g(x)$-clean for a non-zero polynomial $g(x) \in \mathrm{Z}(R)[x]$. For example, the polynomial ring $R[t]$ with R commutative is not $(x^2 - x)$-clean by [36] and is not $(x^n - x)$-clean by [80].*

Next, we consider some types of $(x^2 + cx + d)$-clean rings.

If V is a countable-dimensional vector space over a division ring D, then $\mathrm{End}(V_D)$ is clean by Nicholson and Varadarajan [59]. Furthermore, Camillo and Simón [15] proved that $\mathrm{End}(V_D)$ is $g(x)$-clean provided that $g(x) \in \mathrm{Z}(D)[x]$ has two distinct roots in $\mathrm{Z}(D)$. Recently, this result has been extended to the following.

Example 3.1.11 *[62] Let R be a ring and M_R be a semisimple module over R. If $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ where $a, b \in \mathrm{Z}(R)$ are such that b and $b-a$ are both units in R, then $\mathrm{End}(M_R)$ is $g(x)$-clean.*

Example 3.1.11 (in case of $a = 0$ and $b = 1$) implies that the endomorphism ring of a semisimple module is clean. But it is surprising that Example 3.1.11 does not say more than this. The forthcoming theorem completely determines the relations between $g(x)$-clean rings and clean rings.

Theorem 3.1.12 *Let R be a ring and $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ where $a, b \in \mathrm{Z}(R)$. Then the following hold:*

(i) *R is clean and $b - a \in \mathrm{U}(R)$ iff R is $(x-a)(x-b)$-clean.*

(ii) *If R is clean and $b - a \in \mathrm{U}(R)$, then R is $g(x)$-clean, but not conversely.*

Proof (i) $\Rightarrow$ Let $r \in R$. Since R is clean, $\frac{r-a}{b-a} = e + u$ where $e^2 = e \in R$ and $u \in \mathrm{U}(R)$. Thus, $r = (e(b-a)+a) + u(b-a)$, where $e(b-a)+a$ is a root of $(x-a)(x-b)$ and $u(b-a) \in \mathrm{U}(R)$. Hence, R is $(x-a)(x-b)$-clean.

$\Leftarrow$ Since a is $(x-a)(x-b)$-clean, $a = s + u$ with $(s-a)(s-b) = 0$ and $u \in \mathrm{U}(R)$. Hence $s = b$, which implies that $b - a \in \mathrm{U}(R)$. Let $r \in R$. Since R is $(x-a)(x-b)$-clean, $r(b-a)+a = s+u$, where s is a root of $(x-a)(x-b)$ and $u \in \mathrm{U}(R)$. Thus, $r = \frac{s-a}{b-a} + \frac{u}{b-a}$, where

$$\left(\frac{s-a}{b-a}\right)^2 = \frac{(s-a)(s-b+b-a)}{(b-a)^2} = \frac{(s-a)(b-a)}{(b-a)^2} = \frac{s-a}{b-a}$$

is an idempotent and $\frac{u}{b-a}$ is a unit of R. So R is clean.

(ii) If R is clean and $b - a \in \mathrm{U}(R)$, then R is $g(x)$-clean by (i). By Example 3.1.1, the reversed implication is not true. □

Corollary 3.1.13 *Let R be a ring. Then R is clean iff R is $(x^2 + x)$-clean.*

Proof This is the case of Theorem 3.1.12 (i) when $a = 0$ and $b = -1$. $\square$

Remark 3.1.14 *The equivalence of (x^2+x)-cleanness and cleanness is a ring property. That is, it holds for a ring but it may fail for a single element. For example, $1+1 = 2 \in \mathbb{Z}$ is clean but it is not (x^2+x)-clean in $\mathbb{Z}$ since $\mathbb{Z}$ has only two units 1 and -1.*

For any $n \in \mathbb{N}$, $U_n(R)$ denotes the set of elements of R which can be written as a sum of no more than n units of R [38]. A ring R is called **generated by its units** if $R = \bigcup_{n=1}^{\infty} U_n(R)$. Rings generated by units have been extensively concentrated (see [38, 39, 64]). Here, we use $g(x)$-cleanness to characterize some classes of special ones.

It is an open problem whether or not the clean property of the matrix ring $\mathbb{M}_n(R)$ $(n > 1)$ implies that of R [36]. But for $(x^2 - 2x)$-clean rings, the clean property of R and that of the matrix ring $\mathbb{M}_n(R)$ $(n > 1)$ are equivalent and $(x^2 - 2x)$-clean rings are precisely those rings whose elements can be expressed as the sum of two units and one of them is the square root of 1. More precisely, we have the following result.

Theorem 3.1.15 *Let R be a ring and $m, n, k \in \mathbb{N}$. Then the following are equivalent:*

(i) *R is $(x^2 - 2^n x)$-clean.*

(ii) *R is $(x^2 + 2^n x)$-clean.*

(iii) *R is $(x^2 - 2x)$-clean.*

(iv) *R is $(x^2 + 2x)$-clean.*

(v) *R is $(x^2 - 1)$-clean.*

(vi) *R is clean and $2 \in \mathrm{U}(R)$.*

(vii) *For any $a \in R$, a can be expressed as $a = u + v$ where $u \in \mathrm{U}(R)$ and $v^2 = 1$.*

(viii) *$\mathbb{M}_k(R)$ is $(x^2 - 2x)$-clean.*

(ix) $\mathbb{M}_k(R[[t]])$ *is* $(x^2 - 2x)$*-clean.*

(x) $\mathbb{M}_k\left(\dfrac{R[t]}{(t^m)}\right)$ *is* $(x^2 - 2x)$*-clean.*

Moreover, (i), (ii), (iv), (v), (vi), *and* (vii) *are still equivalent to others if* R *is replaced by* $\mathbb{M}_k(R)$ *or* $\mathbb{M}_k(R[[t]])$ *or* $\mathbb{M}_k\left(\dfrac{R[t]}{(t^m)}\right)$.

Proof (i) $\Rightarrow$ (vi) We prove $2 \in \mathrm{U}(R)$. Suppose $2 \notin \mathrm{U}(R)$. Then $\overline{R} = R/(2^n R) \neq 0$. Let $2^n = s + u$ with $s^2 - 2^n s = 0$ and $u \in \mathrm{U}(R)$. Since $\overline{0} = \overline{2^n} = \overline{s} + \overline{u}$, we have $\overline{s} = -\overline{u} \in \mathrm{U}(\overline{R})$. But $\overline{s}^2 = \overline{s^2} = \overline{2^n s} = \overline{0}$. This is a contradiction. So $2 \in \mathrm{U}(R)$. Then R is clean by (i) of Theorem 3.1.12 with $a = 0$ and $b = 2^n$.

(vi) $\Rightarrow$ (i) By (i) of Theorem 3.1.12, R is $(x^2 - 2^n x)$-clean.

Similarly, we can prove (ii) $\Leftrightarrow$ (vi), (iii) $\Leftrightarrow$ (vi), and (iv) $\Leftrightarrow$ (vi).

(vi) $\Rightarrow$ (vii) Let $a \in R$. By (iii) $\Leftrightarrow$ (vi), $1 - a = s + u$ where $s^2 = 2s$ and $u \in \mathrm{U}(R)$. Then $a = (-u) + (1 - s)$ with $-u \in \mathrm{U}(R)$ and $(1 - s)^2 = 1$ [13, Proposition 10].

(vii) $\Rightarrow$ (vi) Let $a \in R$. By (vii), $1 - a = u + v$ with $u \in \mathrm{U}(R)$ and $v^2 = 1$. Thus $a = (-u) + (1 - v)$ with $-u \in \mathrm{U}(R)$ and $(1 - v)^2 = 2(1 - v)$. By (iii) $\Leftrightarrow$ (vi), (vii) implies (vi).

(v) $\Rightarrow$ (vii) If R is $(x^2 - 1)$-clean, then, for any $a \in R$, there exist $u, v \in \mathrm{U}(R)$ such that $a = u + v$ and $v^2 = 1$.

(vii) $\Rightarrow$ (v) Let $a \in R$. Then a can be expressed as $a = u + v$ with $u, v \in \mathrm{U}(R)$ and $v^2 = 1$. So v is the root of $x^2 - 1$. Hence R is $(x^2 - 1)$-clean.

(viii) $\Leftrightarrow$ (vii) By [32, Corollary 1.6].

(ix) $\Leftrightarrow$ (iii) Since R is $(x^2 - 2x)$-clean iff $R[[t]]$ is $(x^2 - 2x)$-clean by Proposition 3.1.9, the equivalence of (ix) and (iii) follows from (viii) $\Leftrightarrow$ (iii).

(x) $\Leftrightarrow$ (iii) By Proposition 3.1.3, (iii) $\Rightarrow$ (ix) $\Rightarrow$ (x) $\Rightarrow$ (iii).

Others can be proved similarly. □

Remark 3.1.16 *Let $m, k \in \mathbb{N}$. Similar to Theorem 3.1.15, it can be proved that, for a ring R and a fixed $n \in \mathbb{N}$, the following are equivalent:*

(i) *R is $(x^2 - n^m x)$-clean.*

(ii) *R is $(x^2 + n^k x)$-clean.*

(iii) *R is $(x^2 - nx)$-clean.*

(iv) *R is $(x^2 + nx)$-clean.*

(v) *R is a clean ring with $n \in \mathrm{U}(R)$.*

But the other corresponding statements in Theorem 3.1.15 are unknown if $2 \notin \mathrm{U}(R)$.

A module M is called **continuous** if it satisfies (C_1): Every submodule of M is essential in a summand of M and (C_2): If a submodule A of M is isomorphic to a summand of M, then A is a summand of M. A module M is called **discrete** if it satisfies (D_1): For every submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leqslant A$ and $A \cap M_2 << M$ and (D_2): If $A \leqslant M$ such that M/A is isomorphic to a summand of M, then A is a summand of M [51].

Proposition 3.1.17 *Let R be a ring with $n \in \mathrm{U}(R)$. Then, for any continuous or discrete R-module M, the endomorphism ring $\mathrm{End}_R(M)$ is an $(x^2 - nx)$-clean ring.*

Proof Since [12] tells us that every endomorphism ring of continuous or discrete module is clean, Theorem 3.1.12 yields that $\mathrm{End}_R(M)$ is an $(x^2 - nx)$-clean ring since n is invertible in $\mathrm{End}_R(M)$. □

Proposition 3.1.18 *Let X be a strongly zero-dimensional topological space. Then both $\mathbb{M}_k(\mathrm{C}(X))$ and $\mathbb{M}_k(\mathrm{C}^*(X))$ are $(x^2 - nx)$-clean rings for any $n, k \in \mathbb{N}$.*

Proof By [5, Theorem 2.5], $\mathrm{C}(X)$ and $\mathrm{C}^*(X)$ are clean. So they are $(x^2 - nx)$-clean by Theorem 3.1.12 and n is invertible in $\mathrm{C}(X)$ and $\mathrm{C}^*(X)$. Then, by Proposition 3.1.8, $\mathbb{M}_k(\mathrm{C}(X))$ and $\mathbb{M}_k(\mathrm{C}^*(X))$ are $(x^2 - nx)$-clean rings for any $n, k \in \mathbb{N}$. □

Example 3.1.19 *Ehrlich [25] defined the unit regular rings. She proved that if R is a unit regular ring with $2 \in \mathrm{U}(R)$, then every element $rur = r \in R$ with certain $u \in \mathrm{U}(R)$ can be expressed as $r = \frac{2ru-1}{2}u^{-1} + \frac{1}{2}u^{-1}$, that is, $R = U_2(R)$. In fact, for every unit regular ring with $2 \in \mathrm{U}(R)$, the matrix ring $\mathbb{M}_k(R)$, for any $k \in \mathbb{N}$, is an $(x^2 - 2x)$-clean ring by [11] and Proposition 3.1.8.*

Proposition 3.1.20 *Let F be a field with characteristic $\mathrm{char}F = c$, let V be an infinite-dimensional vector space over F, and let R be the subring of $\mathrm{End}_F(V)$ generated by the identity and the finite rank transformations. Then $\mathbb{M}_k(R)$ is an $(x^2 - nx)$-clean ring where $n, k \in \mathbb{N}$ and c does not divide n.*

Proof By [34, Example 5.15], R is a unit regular ring. So by [11], R is clean. Then R is an $(x^2 - nx)$-clean ring since $n \in R$ is a unit. Hence, by Proposition 3.1.8, $\mathbb{M}_k(R)$ is an $(x^2 - nx)$-clean ring for any $n, k \in \mathbb{N}$. □

Proposition 3.1.21 *Let R be a ring with $d \in \mathrm{U}(R)$. If R is $(x^2 + cx + d)$-clean, then $R = U_2(R)$. In particular, if R is $(x^2 + x + 1)$-clean, then $R = U_2(R)$ is $(x^4 - x)$-clean.*

Proof Let $r \in R$. Then $r = s + u$ with $s^2 + cs + d = 0$ and $u \in \mathrm{U}(R)$. So $s(s+c) = (s+c)s = -d \in \mathrm{U}(R)$. Hence $s \in \mathrm{U}(R)$. In other words, $r \in U_2(R)$. Therefore, $R = U_2(R)$. If R is $(x^2 + x + 1)$-clean, then, for any $r \in R$, $r = s + u$ with $s^2 + s + 1 = 0$ and $u \in \mathrm{U}(R)$. This implies that $s^4 - s = s(s-1)(s^2+s+1) = 0$, so R is $(x^4 - x)$-clean. □

A ring R is called **semipotent** if every left (or right) ideal not contained in $J(R)$ contains a nontrivial idempotent (an idempotent that is not 0 or 1 is called a nontrivial idempotent). Every exchange ring is semipotent, so is every clean ring [57]. Notice that any semipotent ring containing no infinite set of orthogonal non-zero idempotents is a semiperfect ring [56]. Since $\mathbb{Z}_{(7)}C_3$ is not a semiperfect ring [73] but is a noetherian ring, it is not semipotent and not exchange. Thus, by Example 3.1.1, an $(x^4 - x)$-clean ring need not be semipotent. By Ye [80, Theorem 5.2], the directly infinite regular ring and direct finite regular ring with 2 invertible constructed by Bergman [37, Examples 1 and 2] are not $(x^n - x)$-clean for every $n \geq 2$.

Proposition 3.1.22 *Let R be a ring with $n \in \mathbb{N}$. Then R is $(ax^{2n} - bx)$-clean iff R is $(ax^{2n} + bx)$-clean.*

Proof $\Rightarrow$ Suppose R is $(ax^{2n} - bx)$-clean. Then, for any $r \in R$, $-r = s + u$, $as^{2n} - bs = 0$, and $u \in \mathrm{U}(R)$. So $r = (-s) + (-u)$ where $a(-s)^{2n} + b(-s) = 0$ and $-u \in \mathrm{U}(R)$. Hence, r is $(ax^{2n} + bx)$-clean. Therefore, R is $(ax^{2n} + bx)$-clean.

$\Leftarrow$ Suppose R is $(ax^{2n} + bx)$-clean. Let $r \in R$. Then there exist s and u such that $-r = s + u$, $as^{2n} + bs = 0$, and $u \in \mathrm{U}(R)$. So $r = (-s) + (-u)$ with $a(-s)^{2n} - b(-s) = 0$ and $-u \in \mathrm{U}(R)$. Hence R is $(ax^{2n} - bx)$-clean. □

By Proposition 3.1.22, we get $\mathbb{Z}_{(7)}C_3$ is also $(x^4 + x)$-clean. For $2n + 1 \in \mathbb{N}$, we do not know whether the $(x^{2n+1} - x)$-cleanness of R is equivalent to $(x^{2n+1} + x)$-cleanness of R.

Lemma 3.1.23 *Let $a \in R$. The following are equivalent for any $n \in \mathbb{N}$:*

(i) $a = a(ua)^n$ *for some* $u \in \mathrm{U}(R)$.

(ii) $a = ve$ *for some* $e^{n+1} = e$ *and some* $v \in \mathrm{U}(R)$.

(iii) $a = fw$ *for some* $f^{n+1} = f$ *and some* $w \in \mathrm{U}(R)$.

Proof (i) $\Rightarrow$ (ii) Suppose (i) holds and let $e = ua$. Then $a = u^{-1}e$ with $e^{n+1} = e$.

(ii) $\Rightarrow$ (iii) Suppose (ii) holds and let $f = vev^{-1}$. Then $a = fv$ with $f^{n+1} = f$.

(iii) $\Rightarrow$ (i) Suppose (iii) holds. Then $(aw^{-1})^{n+1} = f^{n+1} = f = aw^{-1}$. It follows that $a = fw = (aw^{-1})^{n+1}w = a(w^{-1}a)^n$. □

Proposition 3.1.24 *For $2 \leq n \in \mathbb{N}$, let R be an $(x^n - x)$-clean ring. If $a \in R$, then either (i) $a = u + v$ where $u \in \mathrm{U}(R)$ and $v^{n-1} = 1$ or (ii) both aR and Ra contain nontrivial idempotents.*

Proof Write $a = s + u$ where $s^n = s$ and $u \in \mathrm{U}(R)$. Then $as^{n-1} = us^{n-1} + s$. So $a(1 - s^{n-1}) = u(1 - s^{n-1})$. Since $1 - s^{n-1}$ is an idempotent, by Lemma 3.1.23, $u(1 - s^{n-1}) = fw$ where $f^2 = f \in R$ and $w \in \mathrm{U}(R)$. So $f = a(1 - s^{n-1})w^{-1} \in aR$. Suppose (i) does not hold. Then $1 - s^{n-1} \neq 0$. Hence $f \neq 0$. Consequently, aR contains a nontrivial idempotent. Similarly, Ra contains a nontrivial idempotent. □

An element $r \in R$ is called n-**clean** if $r = e + u_1 + \cdots + u_n$ with $e^2 = e \in R$ and $u_i \in \mathrm{U}(R)$ for $1 \leq i \leq n$. And R is called n-**clean** if every element of R is n-clean [74].

Proposition 3.1.25 *Let $n \in \mathbb{N}$. If the ring R is $(x^n - x)$-clean, then R is 2-clean.*

Proof Let $r \in R$. Then $r = s + v$ for some $s^n = s$ and $v \in \mathrm{U}(R)$. Since s is a strongly π-regular element and every strongly π-regular element is strongly clean [58], $s = e + u$ for some $e^2 = e \in R$ and $u \in \mathrm{U}(R)$. So $r = e + u + v$ is 2-clean. Hence, R is 2-clean. □

Remark 3.1.26 *All $(x^2 - x)$-clean rings and $(x^2 + cx + d)$-clean rings with $d \in \mathrm{U}(R)$ discussed above are 2-clean rings.*

3.2 Strongly $g(x)$-clean rings

Following $g(x)$-clean rings, we define strongly $g(x)$-clean rings under Nicholson's suggestion and study some general properties of strongly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings and strongly clean rings.

Definition 3.2.1 *Let $g(x) \in \mathrm{Z}(R)[x]$ be a fixed polynomial. An element $r \in R$ is **strongly** $g(x)$**-clean** if $r = s + u$ with $g(s) = 0$, $u \in \mathrm{U}(R)$, and $su = us$. R is **strongly** $g(x)$**-clean** if every element of R is strongly $g(x)$-clean.*

Strongly clean rings are exactly strongly $(x^2 - x)$-clean rings. However, there are strongly $g(x)$-clean rings which are not strongly clean and vice versa:

Example 3.2.2 *Let R be a commutative local or commutative semiperfect ring with $2 \in \mathrm{U}(R)$. By the proof of [70, Theorem 2.7], RC_3 is strongly (x^6-1)-clean. In particular, $\mathbb{Z}_{(7)}C_3$ is a strongly (x^6-1)-clean ring. Furthermore, by Example 3.1.1, $\mathbb{Z}_{(7)}C_3$ is strongly $(x^4 - x)$-clean but not strongly clean.*

Example 3.2.3 *Let $R = \mathbb{Z}_{(p)}$ and $g(x) = (x-a)(x^2+1) \in \mathrm{Z}(R)[x]$. Then R is strongly clean but by a easy verification we know that R is not strongly $g(x)$-clean. The ring in Example 3.1.2 is strongly clean but not strongly $g(x) = (x+1)(x+c)$-clean.*

However, for some type of polynomials, strong cleanness and strong $g(x)$-cleanness are equivalent.

Theorem 3.2.4 *Let R be a ring and $g(x) \in (x-a)(x-b)\mathrm{Z}(R)[x]$ with $a, b \in \mathrm{Z}(R)$. Then the following hold:*

(i) *R is strongly $(x-a)(x-b)$-clean iff R is strongly clean and $b - a \in \mathrm{U}(R)$.*

(ii) *If R is strongly clean and $b - a \in \mathrm{U}(R)$, then R is strongly $g(x)$-clean, but not conversely.*

Proof (i) $\Leftarrow$ Let $r \in R$. Since R is strongly clean and $b - a \in \mathrm{U}(R)$, $\frac{r-a}{b-a} = e + u$ where $e^2 = e \in R$, $u \in \mathrm{U}(R)$, and $eu = ue$. Thus, $r = (e(b-a)+a) + u(b-a)$ where $u(b-a) \in \mathrm{U}(R)$, $(e(b-a)+a-a)(e(b-a)+a-b) = 0$, and $(e(b-a)+a)u(b-a) = u(b-a)(e(b-a)+a)$. Hence, R is strongly $(x-a)(x-b)$-clean.

$\Rightarrow$ Since a is strongly $(x-a)(x-b)$-clean, there exist $u \in \mathrm{U}(R)$ and $s \in R$ such that $a = s+u$ with $(s-a)(s-b) = 0$ and $su = us$. Hence, $s = b$. So $b-a \in \mathrm{U}(R)$. Let $r \in R$.

Since R is strongly $(x-a)(x-b)$-clean, $r(b-a)+a = s+u$ where $(s-a)(s-b)=0$, $u \in \mathrm{U}(R)$, and $su = us$. Thus, $r = \frac{s-a}{b-a} + \frac{u}{b-a}$ where

$$\left(\frac{s-a}{b-a}\right)^2 = \frac{(s-a)(s-b+b-a)}{(b-a)^2} = \frac{(s-a)(b-a)}{(b-a)^2} = \frac{s-a}{b-a},$$

$\frac{u}{b-a} \in \mathrm{U}(R)$, and $\frac{s-a}{b-a} \cdot \frac{u}{b-a} = \frac{u}{b-a} \cdot \frac{s-a}{b-a}$. So R is strongly clean.

(ii) By (i) and Example 3.2.2. □

Corollary 3.2.5 *For a ring R, R is strongly clean iff R is strongly (x^2+x)-clean.*

Proof It follows from the special case $(a,b)=(0,1)$ of Theorem 3.2.4. □

Remark 3.2.6 *The equivalence of strong (x^2+x)-cleanness and strong cleanness is a ring property since it holds for a ring R but it may fail for a single element. For example, $2 \in \mathbb{Z}$ is strongly clean but not strongly (x^2+x)-clean.*

Example 3.2.7 *If X is strongly zero-dimensional, then $\mathrm{C}(X)$ and $\mathrm{C}^*(X)$ are strongly (x^2-nx)-clean rings for any $n \in \mathbb{N}$ since $\mathrm{C}(X)$ and $\mathrm{C}^*(X)$ are strongly clean [5, 49] and n is invertible in $\mathrm{C}(X)$ and $\mathrm{C}^*(X)$. If X is a P-space, then $\mathbb{M}_k(\mathrm{C}(X))$ is strongly (x^2-nx)-clean for any $n,k \in \mathbb{N}$ because $\mathbb{M}_k(\mathrm{C}(X))$ is strongly clean by Theorem 2.5.1.*

Proposition 3.2.8 *Let $\theta: R \to S$ be a ring epimorphism. If R is strongly $g(x)$-clean, then S is strongly $\theta'(g(x))$-clean.*

Proof Let $g(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathrm{Z}(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in \mathrm{Z}(S)[x]$. For any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is strongly $g(x)$-clean, there exist $t \in R$ and $u \in \mathrm{U}(R)$ such that $r = t+u$ with $g(t)=0$ and $tu = ut$. Then $s = \theta(r) = \theta(t) + \theta(u)$ with $\theta'(g(x))|_{x=\theta(t)} = 0$, $\theta(u) \in \mathrm{U}(S)$, and $\theta(t)\theta(u) = \theta(u)\theta(t)$. So S is strongly $\theta'(g(x))$-clean. □

Corollary 3.2.9 *If R is strongly $g(x)$-clean, then. for any ideal I of R, R/I is strongly $\overline{g}(x)$-clean with $\overline{g}(x) \in \mathrm{Z}(R/I)[x]$.*

Proof This is because R/I is the homomorphic image of R. □

Corollary 3.2.10 *Let R be a ring and $g(x) \in \mathrm{Z}(R)[x]$. If the formal power series ring $R[[t]]$ is strongly $g(x)$-clean, then R is strongly $g(x)$-clean.*

Proof This is because $\theta : R[[t]] \to R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$. □

Proposition 3.2.11 *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i\in I}$ be a family of rings. Then $\prod_{i\in I} R_i$ is strongly $g(x)$-clean if and only if R_i is strongly $g(x)$-clean for each $i \in I$.*

Proof It is clear by the definition and Proposition 3.2.8. □

Proposition 3.2.12 *Let R be a ring, $g(x) \in \mathrm{Z}(R)[x]$, and $1 < n \in \mathbb{N}$. If $\mathbb{T}_n(R)$ is strongly $g(x)$-clean, then R is strongly $g(x)$-clean.*

Proof For any $a \in R$, let A be the diagonal matrix $\mathrm{diag}(a, \cdots, a)$. Then by a direct computation, strong $g(x)$-cleanness of A implies strong $g(x)$-cleanness of a. □

For strongly clean rings, the authors in [66, 20, 18] proved that if R is a strongly clean ring and $e^2 = e \in R$, then the corner ring eRe is strongly clean. For strongly $g(x)$-clean rings, we have the following result.

Theorem 3.2.13 *Let R be a strongly $(x-a)(x-b)$-clean ring with $a, b \in \mathrm{Z}(R)$. Then, for any $e^2 = e \in R, eRe$ is strongly $(x-ea)(x-eb)$-clean. In particular, if $g(x) \in (x-ea)(x-eb)\mathrm{Z}(R)[x]$ and R is strongly $(x-a)(x-b)$-clean with $a, b \in \mathrm{Z}(R)$, then eRe is strongly $g(x)$-clean.*

Proof By Theorem 3.2.4, R is strongly $(x-a)(x-b)$-clean iff R is strongly clean and $b-a \in \mathrm{U}(R)$. If R is strongly clean, then eRe is strongly clean. Again by Theorem 3.2.4, eRe is strongly $(x-ea)(x-eb)$-clean. □

However, generally speaking, the strongly $g(x)$-clean property is not a Morita invariant: When $g(x) = (x-a)(x-b)$ where $a, b \in \mathrm{Z}(R)$ with $b-a \in \mathrm{U}(R)$, the matrix ring over the local ring $\mathbb{Z}_{(p)}$ is not strongly clean [18] and not strongly $g(x)$-clean.

We use strong $g(x)$-cleanness to characterize some special rings generated by units in which every element can be written as the sum of a unit and a square root of 1 which commute.

Theorem 3.2.14 *Let R be a ring and $n \in \mathbb{N}$. Then the following are equivalent:*

(i) *R is strongly $(x^2 - 2^n x)$-clean.*

(ii) *R is strongly (x^2-1)-clean.*

(iii) *R is strongly clean and $2 \in \mathrm{U}(R)$.*

(iv) *$R = U_2(R)$ and for any $a \in R$, a can be expressed as $a = u + v$ with some $u, v \in \mathrm{U}(R)$, $uv = vu$, and $v^2 = 1$.*

Proof (i) ⇒ (iii) To prove $2 \in \mathrm{U}(R)$. Suppose $2 \notin \mathrm{U}(R)$, then $\overline{R} = R/(2^n R) \neq 0$. Let $2^n = s + u$ with $s^2 - 2^n s = 0, u \in \mathrm{U}(R)$, and $su = us$. $\overline{0} = \overline{2^n} = \overline{s} + \overline{u}$ implies that $\overline{s} = -\overline{u} \in \mathrm{U}(\overline{R})$. But $\overline{s}^2 = \overline{s^2} = \overline{2^n s} = \overline{0}$, a contradiction. So $2 \in \mathrm{U}(R)$. Let $a = 0$ and $b = 2^n$. Then, by (i) of Theorem 3.2.4, R is strongly clean.

(iii) ⇒ (i) By (i) of Theorem 3.2.4, R is strongly $(x^2 - 2^n x)$-clean.

(iii) ⇒ (iv) Let $a \in R$. By (i) ⇔ (iii), let $n = 1$. Then $1 - a = s + u$ where $s^2 = 2s, u \in \mathrm{U}(R)$, and $su = us$. Then $a = (1-s) + (-u)$ with $(1-s)^2 = 1$, $-u \in \mathrm{U}(R)$, and $(1-s)(-u) = (-u)(1-s)$.

(iv) $\Rightarrow$ (iii) Let $a \in R$. By (iv), $1 - a = u + v$ where $u \in \mathrm{U}(R), v^2 = 1$, and $uv = vu$. Thus, $a = (-u) + (1 - v)$ with $-u \in \mathrm{U}(R), (1 - v)^2 = 2(1 - v)$, and $(-u)(1 - v) = (1 - v)(-u)$. By (i) $\Leftrightarrow$ (iii) and $n = 1$, we proved that (iv) implies (iii).

(ii) $\Rightarrow$ (iv) If R is strongly $(x^2 - 1)$-clean, then, for any $a \in R$, there exist $u, v \in \mathrm{U}(R)$ such that $a = u + v$ with $v^2 = 1$ and $uv = vu$.

(iv) $\Rightarrow$ (ii) Let $a \in R$. Then a can be expressed as $a = u + v$ with $u, v \in \mathrm{U}(R), v^2 = 1$, and $uv = vu$. So v is the root of $x^2 - 1$. Hence R is strongly $(x^2 - 1)$-clean. □

Example 3.2.15 *Rings in Example 3.2.7 are strongly $(x^2 - nx)$-clean. In particular, they are strongly $(x^2 - 2^n x)$-clean rings in which every element can be written as the sum of a unit and a square root of 1 which commute.*

A ring R is called **locally artinian** provided that every finitely generated subring of R is artinian. Let S be a subring of R. R is called **locally artinian over** S if every finitely generated subring $S[a_1, \cdots, a_n]$ is artinian.

Let R be a commutative ring. A chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ is said to be of **length** r. Let $M = \{r : \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ is a chain of prime ideals of $R\}$. If $l = \mathrm{Sup}M$, then we say that R has **Krull dimension** l. If $l = 0$, then R is a 0-dimensional ring, that is, every prime ideal of R is maximal. If $l = 1$, then R is a 1-dimensional ring. The ring of integers $\mathbb{Z}$ is 1-dimensional because $0 \subset p\mathbb{Z}$ is the only chain of prime ideals of length 1 where p is any prime number. The ring in Proposition 3.2.17 is 0-dimensional.

Example 3.2.16 *Let R be the ring in Proposition 3.1.20. Then R is strongly clean [66, Example 2.7]. In fact, R is locally artinian. So the matrix ring $\mathbb{M}_k(R)$ is strongly $(x^2 - nx)$-clean with $\mathrm{char}F \not| n$. If $\mathrm{char}F \neq 2$, then every element in the matrix ring can be written as the sum of a unit and a square root of 1 which commute.*

Proposition 3.2.17 *Let $A = F[x_1, x_2, \cdots]$ be the polynomial ring in a countably infinite set of indeterminates $(x_1, x_2, \cdots)$ over a field F, and let $I = < x_1^{k_1}, x_2^{k_2}, x_3^{k_3}, \ldots >$ with $k_i \in \mathbb{N}$. Then $R = A/I$ is a local ring of dimension 0 which is not noetherian. But R is locally artinian. So the matrix ring $\mathbb{M}_k(R)$ is strongly $(x^2 - nx)$-clean with* char$F \not| n$. *If* char$F \neq 2$, *then every element in the matrix ring can be written as the sum of a unit and a square root of 1 which commute.*

Proof Note that an ideal $\mathfrak{m}$ is maximal in R iff $R/\mathfrak{m}$ is a field. Let $J = \frac{<x_1, x_2, \cdots>}{<x_1^{k_1}, x_2^{k_2}, \cdots>}$. J is maximal because

$$R/J = \frac{\frac{F[x_1, x_2, \cdots]}{<x_1^{k_1}, x_2^{k_2}, \cdots>}}{\frac{<x_1, x_2, \cdots>}{<x_1^{k_1}, x_2^{k_2}, \cdots>}} \cong \frac{F[x_1, x_2, \cdots]}{< x_1, x_2, \cdots >} \cong F$$

which is a field. Recall that the Jacobson radical $J(R)$ is the intersection of all maximal ideals of R. So $J(R) \subseteq J$. Let $\overline{x_i} = x_i + I$. Note that $\overline{x_i}^{k_i} = 0$ and J is generated by $\overline{x_i}$. So J is a nil ideal. Jacobson radical contains all nil ideals, so $J \subseteq J(R)$. Therefore, $J(R) = J$. Hence R is local with $J(R) = \frac{<x_1, x_2, \cdots>}{<x_1^{k_1}, x_2^{k_2}, \cdots>}$ being the maximal ideal by [1, Proposition 15.15]. Note also that every prime ideal should contain all nilpotent elements of R, that is, every prime ideal should contain J. So, using the fact that J is maximal, we find that the chain of prime ideals in R is only J itself and so R is 0-dimensional. But R is not noetherian because

$$< \overline{x_1} > \subset < \overline{x_1}, \overline{x_2} > \subset \cdots < \overline{x_1}, \cdots, \overline{x_i} > \subset < \overline{x_1}, \cdots, \overline{x_{i+1}} > \subset \cdots$$

is a strictly increasing chain. Furthermore, $S = F[\overline{x_{i_1}}, \overline{x_{i_2}}, \cdots \overline{x_{i_t}}]$ is artinian because S is noetherian with nilpotent Jacobson radical $J(S) = < \overline{x_{i_1}}, \overline{x_{i_2}}, \cdots \overline{x_{i_t}} >$ by [1, Theorem 15.20]. Now for any $A = (a_{ij}) \in \mathbb{M}_k(R)$, there are finitely many $\overline{x_t}$, say, $t = 1, \cdots, m$, such that $a_{ij} \in F[\overline{x_1}, \cdots, \overline{x_m}]$. So $A \in \mathbb{M}_k(F[\overline{x_1}, \cdots, \overline{x_m}])$. Note that the property of being artinian is a Morita invariant for a ring. So $\mathbb{M}_k(F[\overline{x_1}, \cdots, \overline{x_m}])$ is artinian. Hence A is strongly clean in $\mathbb{M}_k(F[\overline{x_1}, \cdots, \overline{x_m}])$. Therefore, A is strongly clean in $\mathbb{M}_k(R)$. By Theorem 3.2.4, we are done. $\square$

Proposition 3.2.18 *Let R be a ring with $c, d \in \mathrm{Z}(R)$ and $d \in \mathrm{U}(R)$. If R is strongly $(x^2 + cx + d)$-clean, then $R = U_2(R)$. In particular, if R is strongly $(x^2 + x + 1)$-clean, then $R = U_2(R)$ is strongly $(x^4 - x)$-clean with every element being the sum of a unit and a cubic root of 1 which commute.*

Proof The first statement is trivial. Let $r \in R$. Then $r = s + u$ with $s^2 + s + 1 = 0$, $u \in \mathrm{U}(R)$, and $su = us$. So $s^4 - s = 0$. Thus, R is strongly $(x^4 - x)$-clean. Moreover, every element in strongly $(x^2 + x + 1)$-clean ring R can be written as the sum of a unit and a cubic root of 1 which commute. $\square$

Proposition 3.2.19 *Let R be a strongly $(x^n - x)$-clean ring where $n \geq 2$ and $a \in R$. Then either (i) $a = u + v$ where $u \in \mathrm{U}(R), v^{n-1} = 1$, and $uv = vu$ or (ii) both aR and Ra contain nontrivial idempotents.*

Proof Since R is strongly $(x^n - x)$-clean, $a = s + u$ with $u \in \mathrm{U}(R)$, $s^n = s$, and $su = us$. Then $s^{n-1}a = s^{n-1}u + s$. So $(1 - s^{n-1})a = (1 - s^{n-1})u$. Since $1 - s^{n-1}$ is an idempotent, by Lemma 3.1.23, $(1 - s^{n-1})u = vg$ where $v \in \mathrm{U}(R)$ and $g^2 = g \in R$. So $g = v^{-1}(1 - s^{n-1})a \in Ra$. Suppose (i) does not hold, then $1 - s^{n-1} \neq 0$, which implies that $g \neq 0$. Thus, Ra contains a nontrivial idempotent. Similarly, aR contains a nontrivial idempotent. $\square$

Finally, we give a property which has nothing to do with rings generated by units but has a close relation with strongly $(x^n - x)$-clean rings.

Proposition 3.2.20 *Let R be a ring and $n \in \mathbb{N}$. Then R is strongly $(ax^{2n} - bx)$-clean iff R is strongly $(ax^{2n} + bx)$-clean.*

Proof $\Rightarrow$ Suppose R is strongly $(ax^{2n} - bx)$-clean. Then, for any $r \in R$, $-r = s + u$ with $as^{2n} - bs = 0, u \in \mathrm{U}(R)$, and $su = us$. So $r = (-s) + (-u)$ where $a(-s)^{2n} + b(-s) = 0, -u \in \mathrm{U}(R)$, and $(-s)(-u) = (-u)(-s)$. Hence, r is strongly $(ax^{2n} + bx)$-clean.

Therefore, R is strongly $(ax^{2n} + bx)$-clean.

$\Leftarrow$ Suppose R is strongly $(ax^{2n} + bx)$-clean. Let $r \in R$. Then there exist s and u such that $-r = s + u$, $as^{2n} + bs = 0$, $u \in \mathrm{U}(R)$, and $su = us$. So $r = (-s) + (-u)$ satisfies $a(-s)^{2n} - b(-s) = 0$, $-u \in \mathrm{U}(R)$, and $(-s)(-u) = (-u)(-s)$. Hence, R is strongly $(ax^{2n} - bx)$-clean. □

Remark 3.2.21 *For $2n+1 \in \mathbb{N}$, we do not know whether the strong $(x^{2n+1} - x)$-cleanness of R is equivalent to the strong $(x^{2n+1} + x)$-cleanness of R.*

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